18.306: Advanced Partial Differential Equations with Applications
M.I.T. Department of Mathematics **Spring 2004**
Practice Set 5 **Handed out**: Friday, 05/07/04
Due: NEVER

You are NOT required to return solutions. None of these problems will be given in Quiz 2. The description of the references given in parentheses can be found in the Bibliography for 18.306.

41. (a) Show that if the relation $\omega = W(k)$ for a wave gives a phase velocity $v_{ph}(k)$ equal to the group velocity $v_g(k)$, the wave cannot be called dispersive.

    (b) Explain in detail why a linear wave with $W(k) = ak + b$, a, $b \neq 0$: constants, cannot be dispersive although $v_{ph} \neq v_g$. Are any discontinuities in data allowed to propagate in this case? Why?

42. The nonlinear Schrödinger equation (NLSE), which governs the non-relativistic motion of each atom in an interacting gas, reads

$$i\Psi_t(x,t) = -\Psi_{xx}(x,t) + a\,|\Psi(x,t)|^2\,\Psi(x,t) - a\Psi(x,t), \quad -\infty < x, t < +\infty;\ a = \text{const.} > 0.$$

    (a) Substitute $\Psi(x,t) = A(x,t)e^{i\theta(x,t)}$, where A and θ are real functions, and derive two coupled PDE for A and θ.

    (b) From (a), solve for θ by assuming that A is slowly varying in both space and time (i.e., take $A \approx$ const.) Derive the dispersion relation $\omega = \omega(k)$ for $A \approx 1$, where $k \equiv \theta_x$ and $\omega \equiv -\theta_t$. Justify that your solution is a dispersive wave. What is the group velocity?

43. (Levine, Chap. 5, Prob. 5, p. 62.) The gravity-driven motion of a thin fluid film on an inclined plane is governed approximately by the PDE: $h_t + (g \sin\phi/\nu)h^2\, h_x = 0$. In this equation, $h = h(x,t)$ is the fluid thickness (height from incline) as a function of the coordinate x along the plane and the time t $(t > 0)$, ϕ is the angle of the incline, g is the gravity acceleration, and ν is the viscosity.

    (a) Show that the general solution to the given PDE is given implicitly by the equation $h = F(x - g\sin\phi\, h^2 t/\nu)$, where F is an arbitrary function.
    (b) Obtain a particular solution to the PDE that satisfies the initial condition $h(x,0) = \alpha x$, for $0 < x < L$, and $h(x,0) = 0$ otherwise. From this particular solution, derive the approximate formula $h(x,t) \sim (g\sin\phi/\nu)^{-1/2}\,(x/t)^{1/2}$ for $t \to \infty$ and $x > 0$, which has a similarity form.

    Now re-derive the formula in (b) via a similarity solution in 2 alternative ways:
    (c) Show that the given PDE is invariant under the stretching transformations $x' = \kappa x$, $t' = \kappa t$ and $h' = h$, where $\kappa > 0$ is an arbitrary parameter. Hence, try a similarity solution $h = T(t)\, g(\eta)$ where $\eta = x/t$. Find $T(t)$ and $g(\eta)$ via substitution into the PDE.
    (d) Look for solutions $h = c\, x^\beta t^\gamma$. Find the constants c, β and γ by substitution into the PDE.

44. Consider the usual wave equation, $u_{tt} - c^2 u_{xx} = 0$.

(a) Dimensional arguments suggest a similarity solution of the form $u = f(\xi)$, where $\xi = \frac{x}{ct}$. Find and solve the ODE for $f(\xi)$.

(b) Substitute $u = x^\gamma f(\xi)$ into the PDE, where $\xi = \frac{x}{ct}$ and γ is real. Find an ODE for $f(\xi)$.

45. (Debnath, Prob. 8.14.7, pp. 327, 328.) The Prandtl-Blasius problem for a flat plate in a uniformly moving fluid is described by the PDE system

$$uu_x + vu_y = \nu u_{yy}, \qquad u_x + v_y = 0,$$

where u and v are the fluid velocity components in the x and y directions and ν is a positive constant.

(a) Introduce the stream function ψ by $(u, v) = (\psi_y, -\psi_x)$ and derive the PDE satisfied by ψ.

(b) In (a) try a similarity solution of form $\psi = x^{1/2}\, g(\eta)$ where $\eta = y\, x^{-1/2}$. Derive an ODE for g.

(c) Can you justify in any way, mathematical and/or physical, the form of solution suggested in (b)?

46. (Levine, Chap. 5, Prob. 3, pp. 60, 61.) Consider the PDE $u_t = \nu(u_{rr} + r^{-1}u_r - r^{-2}u)$ with variable coefficients, where $r > 0, t > 0$, and $\nu > 0$ is a constant; $u = u(r,t)$ is an axisymmetric function in two space dimensions.

(a) Apply stretching transformations (ST) and find a relation among the exponents so that the PDE is invariant under ST.

(b) By virtue of (a), try a similarity solution of the form $u = r^\zeta f(\eta)$. Define the similarity variable η. Derive an ODE for $f(\eta)$ for any admissible value of real ζ.

(c) In (b), choose a negative ζ so that the ODE for f contains only f'' and f' terms and no f term. Solve the resulting ODE by applying the conditions that $u(0,t)$: finite and $\lim_{t \to 0^+}(ru) = k$: given.

(d) Define $w(r,t) \equiv u_r + r^{-1}u$. Derive a PDE for w. Try a solution of this PDE in the similarity form $w = t^{-1}g(\eta)$, where η is the similarity variable of part (b) above. Compare the results for g with the result of part (c).

47. (a) (Drazin & Johnson, Prob. Q2.11, p. 35.) Consider the nonlinear Schrödinger equation, $iu_t + u_{xx} + \nu\, u|u|^2 = 0$, where $u = u(x,t)$ and $\nu > 0$, and apply the ST $\tilde{x} = \kappa^\alpha x, \tilde{t} = \kappa^\beta t$ and $\tilde{u} = \kappa^\gamma u$. Find relations among α, β and γ so that the PDE is invariant under ST. Set $\alpha = 1$: Try a similarity solution $u = t^\zeta f(\eta)$. Give ζ and η and derive an ODE for f.

(b) (Drazin & Johnson, Prob. Q2.8, p. 34.) Determine μ and λ so that $u(x,t) = t^\mu f(xt^\lambda)$ is a (similarity) solution of Burgers' equation,
$u_t + uu_x = u_{xx}$. Find and solve the ODE for f with the condition that $f \to 0$ as $x \to \infty$ and $f(0) = -2\pi^{-1/2}$.

48. (Carrier & Pearson, Prob. 16.5.1, p. 313.) The following problem arises in the flow of a viscous heat-conducting fluid along a channel, and illustrates a situation in which it is useful to change

the scale of the dependent, as well as the independent, variable. Let $u = u(x, y)$ satisfy the PDE $\epsilon u_{yy} - (1 - y^2)u_x + 4y^2 = 0$ in the region $x > 0, -1 < y < 1$, where $0 < \epsilon \ll 1; u$ obeys the conditions $u(0, y) = 0$ and $u(x, \pm 1) = 1$.

(a) Find the solution $u = u_0$ for $\epsilon = 0$ that satisfies the first one of the conditions, and explain why this is not acceptable.

(b) Introduce a boundary layer near $y = -1$ by setting $\zeta = \epsilon^{-\alpha}(1 + y)$. Show that $u_0 = O(\epsilon^{-\alpha})$ near the boundary layer. This suggests writing $u = \epsilon^{-\alpha}\psi(x, \zeta)$ inside the boundary layer. Derive a PDE for ψ and determine α. Give suitable conditions for ψ.