**18.306:** Advanced Partial Differential Equations with Applications
**M.I.T. Department of Mathematics** **Spring 2004**
**Homework 2** **Handed out**: Wednesday, 02/25/04
**Due:** Wednesday, 03/10/04

The description of the references in parentheses can be found in the Bibliography for 18.306.

4. (Carrier & Pearson, Prob. 6.4.4, p. 98) In a rotating-fluid problem (spin-up) it is required that a function $u(r,\tau)$ be determined in the region $0 < r < a,\ \tau > 0$, where $r$ is the polar distance and the dimensionless $\tau$ measures time (properly scaled); $u$ satisfies

$$u_\tau - \left(1 - \frac{u}{r}\right)(ru)_r = 0, \quad u(r,0) = 0,\ u(a,\tau) = a.$$

Use the characteristics of the given PDE to show that the solution of this problem is described by: $u = \frac{re^{2\tau} - (a^2/r)}{e^{2\tau}-1}$ for $r \geq ae^{-\tau}$, and $u = 0$ for $r < ae^{-\tau}$. Give a rough sketch of the characteristics and discuss the behavior of the solution near the point $(r,\tau) = (a,0)$.
**Hint:** Set $v = ru$.

5. (Carrier & Pearson, Prob. 13.10.2, p. 255) In the two-dimensional steady rotational flow of an ideal gas, the two velocity components $u$ and $v$, and the gas density $\rho$ satisfy the PDE system

$$(\rho u)_x + (\rho v)_y = 0, \quad \rho(u_x u + u_y v) = -c^2\,\rho_x, \quad \rho(v_x u + v_y v) = -c^2\,\rho_y,$$

where $c$ is the (local) sound velocity. (What does the first PDE mean physically?)

(a) Give an inequality that $u, v$ and $c$ should satisfy so that this PDE system has real characteristics. Determine the characteristic directions.

(b) For part (a), give the relations among $u, v$ and $\rho$, or their variations, along the characteristics.

6. (a) An electrical signal described by the function $u(x,t)$ is transmitted in the $x$ direction. If $u$ satisfies the wave equation, $u_{tt} - c^2 u_{xx} = 0$, find $u$ in the region $x > 0$, $t > 0$, with the conditions

$$\begin{cases} u(x,0) = q(x),\ u_t(x,0) = 0, \\ u(0,t) = p(t), \end{cases}$$

where $q(x)$ and $p(t)$ are known functions.

(b) Indicate with the help of a sketch the region of the $xt$-plane affected by the condition along $x = 0$ and the condition along $t = 0$. Can you interpret your solution in terms of the motion of a stretched string?

7. The (properly nondimensionalized) displacement $u(x,t)$ of a damped vibrating string with harmonic forcing $F(x,t) = f(x)e^{i\Omega t}, \Omega > 0$, satisfies the PDE

$$u_{xx} - u_{tt} - \epsilon u_t = F(x,t), \quad -\infty < x < \infty,\ \epsilon > 0.$$

Find a particular solution of this PDE in the form $u(x,t) = \phi(x)e^{i\Omega t}$, with $\phi(x) \to 0$ as $|x| \to \infty$. (What is the condition that $f(x)$ must satisfy?) Explain what happens if $\epsilon \ll \Omega$.