Apply the idea:

$$u = \left(\frac{a}{\sqrt{\nu t}}\right)^{\beta} h(\xi)$$

$$\int_{-\infty}^{+\infty} dx\, u = \int_0^a dx\, f(x) = \int_{-\infty}^{+\infty} dx \left(\frac{a}{\sqrt{\nu t}}\right)^{\beta} h(\xi)$$

$$= \sqrt{\nu t} \int_{-\infty}^{+\infty} d\xi \left(\frac{a}{\sqrt{\nu t}}\right)^{\beta} h(\xi)$$

$$= \sqrt{\nu t} \left(\frac{a}{\sqrt{\nu t}}\right)^{\beta} \underbrace{\int_{-\infty}^{+\infty} d\xi\, h(\xi)}_{\text{number} = C}$$

$$\int_0^a dx\, f(x) = \frac{a^{\beta}}{(\sqrt{\nu t})^{\beta - 1}}\, C$$

do not depend on t — should not dependent on t $\Rightarrow \beta = 1$

April 28, 2004 — Lecture 23

Extra lecture: FRI 4-5:30

Return: Hmwk 5 (Place in envelope)

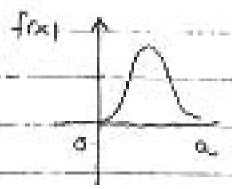
Pick up: Graded Hmwk 4

New Hmwk 6

__Review__: Example PDE

$$\begin{cases} u_t = \nu u_{xx} \\ u(x, t=0) = f(x) \\ u \to 0 \text{ "fast" as } |x| \to \infty \end{cases}$$

From exact solution:



$$u(x,t) \approx \frac{e^{-x^2/4\nu t}}{\sqrt{4\pi \nu t}} \int_0^a dx\, f(x)$$

conditions $\dfrac{a}{\sqrt{\nu t}} \ll 1$ $\quad \dfrac{x}{\sqrt{\nu t}} \geq O(1)$

Get u under same conditions without solving PDE exactly.

$\boxed{\text{Method A :}}$ Dimensional Analysis

(i) $\dfrac{a}{\sqrt{\nu t}}$, $\dfrac{x}{\sqrt{\nu t}}$, ... : independent non-dimensional parameters from x, t

(ii) $u = g\left(\dfrac{a}{\sqrt{\nu t}}, \dfrac{x}{\sqrt{\nu t}}\right) \overset{a/\sqrt{\nu t} \to 0}{\simeq} \left(\dfrac{a}{\sqrt{\nu t}}\right)^{\beta} h\left(\dfrac{x}{\sqrt{\nu t}}\right)$ → ξ : similarity variable

→ β : exponent to be found

- PDE ⇒ ODE for $h(\xi)$

$$h''(\xi) + \frac{\xi}{2} h'(\xi) + \frac{\beta}{2} h(\xi) = 0$$

- "Conservation of mass" ⇒ β : $\displaystyle\int_{-\infty}^{+\infty} dx\, u(x,t) = \text{const} \Rightarrow \beta = 1$
  $(u, u_x \to 0 \quad |x| \to \infty)$

$u(x,t) \simeq \dfrac{a}{\sqrt{\nu t}} h(\xi)$ approximate solution
everything has to be consistent with this form

Solve for $h(\xi)$:

ODE $\begin{cases} h''(\xi) + \dfrac{\xi}{2} h'(\xi) + \dfrac{1}{2} h(\xi) = 0 \\ h(\xi) \text{ has to go to 0 fast as } |\xi| \to \infty \end{cases}$

We look at the solution at later times, we don't see the fine details $\dfrac{a}{\sqrt{\nu t}} \ll 1 \quad \dfrac{x}{\sqrt{\nu t}} \geq O(1)$

$$\text{const} = \int_{-\infty}^{+\infty} dx\, u(x,t) = \cancel{\sqrt{\nu t}} \int_{-\infty}^{+\infty} d\xi \frac{a}{\cancel{\sqrt{\nu t}}} h(\xi) = \int_{-\infty}^{+\infty} dx\, \underbrace{f(x)}_{u|_{t=0}}$$

$x \to \xi$, $dx = \sqrt{\nu t}\, d\xi$

$$\Rightarrow \boxed{\int_{-\infty}^{+\infty} d\xi\, h(\xi) = \frac{1}{a} \int_{-\infty}^{+\infty} dx\, f(x)} \qquad \text{(second condition)}$$

Solve ODE : $h''(\xi) + \dfrac{\xi}{2} h'(\xi) + \dfrac{1}{2} h(\xi) = 0$

$$h''(\xi) + \frac{1}{2}\frac{d}{d\xi}\left[\xi h(\xi)\right] = 0$$

integrate in $(-\infty, \xi)$ $\quad h'(\xi) - \underbrace{h'(-\infty)}_{0} + \dfrac{1}{2}\left[\xi h(\xi) - \underbrace{[\xi h(\xi)]_{-\infty}}_{0}\right] = C_1 = 0$

$$h'(\xi) + \frac{1}{2}\xi h(\xi) = 0 \Rightarrow h(\xi) = C_2 e^{-\xi^2/4}$$

Apply the integral constraint...

$$\int_{-\infty}^{+\infty} d\xi\, h(\xi) = \frac{1}{a}\int_{-\infty}^{+\infty} dx\, f(x) = C_2 \underbrace{\int_{-\infty}^{+\infty} d\xi\, e^{-\xi^2/4}}_{2\sqrt{\pi}}$$

$$\Rightarrow C_2 = \frac{1}{\sqrt{4\pi}}\,\frac{1}{a}\int_{-\infty}^{+\infty} dx\, f(x)$$

$$u(x,t) \simeq \frac{a}{\sqrt{\nu t}}\, h(\xi) = \frac{a}{\sqrt{4\pi\nu t}}\,\frac{1}{a}\int_{-\infty\to 0}^{+\infty\to a} dx\, f(x)\, e^{-\frac{x^2}{4\nu t}}$$

same as $\underbrace{\ }_{\text{from}}$ exact solution.

This becomes an exact solution only if $f(x) = \delta(x)$.

$\boxed{\text{Method B :}}$ Group theory based

Invariance of PDE under " stretching transformations".

" stretching transformation" $\begin{cases} x' = \lambda^{\alpha} x \\ t' = \lambda^{\beta} t \\ u' = \lambda^{\gamma} u \end{cases}$ λ: arbitrary > 0 (const)

α, β, γ : real to be found

(i) PDE in (x', t', u') ; demand that the PDE is the same form as the original PDE.

$$\frac{\partial u'}{\partial t'} - \nu \frac{\partial^2 u'}{\partial x'^2} = \lambda^{\gamma-\beta}\frac{\partial u}{\partial t} - \nu\lambda^{\gamma-2\alpha}\frac{\partial^2 u}{\partial x^2}$$

change notation : $\gamma \to \delta$, $\beta \to \gamma$

$$= \lambda^{\delta-\gamma}\frac{\partial u}{\partial t} - \nu\lambda^{\delta-2\alpha}\frac{\partial^2 u}{\partial x^2} = 0$$

the PDE in '
↳ we want to look the same as the PDE

$$\Rightarrow \delta - \gamma = \delta - 2\alpha \Rightarrow \boxed{\gamma = 2\alpha}$$

(ii) Identify invariant quantities in the 2 systems

$$x' = \lambda^{\alpha} x, \quad t' = \lambda^{2\alpha} t, \quad u' = \lambda^{\delta} u$$

$$\lambda = \left(\frac{x'}{x}\right)^{1/\alpha} = \left(\frac{t'}{t}\right)^{1/2\alpha} = \left(\frac{u'}{u}\right)^{1/\delta}$$

$$\frac{x'}{x} = \left(\frac{t'}{t}\right)^{1/2}$$

I $\boxed{\frac{x'}{\sqrt{t'}} = \frac{x}{\sqrt{t}}}$ INVARIANT OF THE PROBLEM UNDER STRETCHING TRANSFORMATION

II $\frac{u'}{u} = \left(\frac{t'}{t}\right)^{\frac{\delta}{2\alpha}} \Rightarrow \frac{u'}{(t')^{\delta/2\alpha}} = \frac{u}{(t)^{\delta/2\alpha}}$

$\frac{\delta}{\alpha} = -\beta$ arbitrary $\boxed{\frac{u'}{(t')^{-\beta}} = \frac{u}{(t)^{-\beta}}}$ INVARIANT UNDER ST

(iii) "Construct" a solution, similarity solution, that respects invariants under transformation $(x,t,u) \Rightarrow (x',t',u')$

$$\frac{u}{t^{-\beta}} = h_1\left(\frac{x}{\sqrt{t}}\right) \Longleftrightarrow u = \left(\frac{a}{\sqrt{\nu t}}\right)^{\beta} h(\xi), \quad \xi = \frac{x}{\sqrt{\nu t}}$$

same as what we had before

? where is the condition $t \to \infty$, $\left(\frac{a}{\sqrt{\nu t}}\right) \ll 1$

the respect of the invariants by the solution u is not necessary.
it happens under some assumptions $\Rightarrow$ Method A : for $t \gg 1$
Method B includes Method A

Example 2 : Nonlinear diffusion
why systems behave like scaling $\rightarrow$ related to statistic, microscopical

Recall : $\frac{\partial \rho}{\partial t} + \frac{\partial q}{\partial x} = 0$; simple case of diffusion $q = -\nu \rho_x$

- $\nu = \text{const} \Rightarrow \rho_t = \nu \rho_{xx}$ (linear diffusion equation)
- $\nu \neq \text{const}$ e.g. $\nu = g(\rho)$ thermal conduction in solids, filtration theories

PDE: $\frac{\partial \rho}{\partial t} = \frac{\partial}{\partial x}\left(g(\rho)\frac{\partial \rho}{\partial x}\right)$: nonlinear diffusion equation

special case : $g(\rho) = K\rho^n$ $K > 0$ n: real

$$\frac{\partial \rho}{\partial t} = \kappa \frac{\partial}{\partial x}\left(\rho^n \frac{\partial \rho}{\partial x}\right)$$ NOT exactly solvable for $n \neq 0$

Nice sandwich:

analytical thinking
computer
analytical thinking

Try similarity solution:

PDE: $\begin{cases} \dfrac{\partial \rho}{\partial t} = \kappa \dfrac{\partial}{\partial x}\left(\rho^n \dfrac{\partial}{\partial x}\rho\right) \\ \rho(x, t=0) = f(x) \\ \rho \to 0 \text{ fast as } |x| \to +\infty \end{cases}$ Take $n=1$

ρ: continuous in x

Method A

Quantities:	ρ	x	t	κ	m
Dimensions:	$[\rho]$	L	T	$\frac{L^2}{T}\frac{1}{[\rho]}$	$L[\rho]$

what matters for a similarity solution is the integral of the IC:

$$\int_{-\infty}^{+\infty} dx\, \rho(x,t) = \int_{-\infty}^{+\infty} f(x)\, dx = m = \text{const}$$

(You will never use the IC as such, only its integral)

Identify non-dimensional parameters:

$[\kappa] = \frac{[x^2]}{[t]} \frac{1}{[\rho]} \Rightarrow \frac{x^2}{t} \frac{1}{\rho\kappa}$ non-dimensional

$[m] = [x][\rho] \Rightarrow \frac{[m]}{[x]} = [\rho] \Rightarrow \frac{x^3}{m\kappa t}$ non-dimensional

$\boxed{\dfrac{x}{(m\kappa t)^{1/3}}}$ non-dimensional

$[x] = \frac{[m]}{[\rho]} \Rightarrow \frac{m^2}{\rho^2}\frac{1}{t}\frac{1}{\rho\kappa} = \frac{m^2}{\rho^3 \kappa t}$ non-dimensional

$\boxed{\dfrac{\rho}{(m^2/\kappa t)^{1/3}}}$ non-dimensional

Non-dimensional quantities: $\frac{x}{(m\kappa t)^{1/3}}$, $\rho\left(\frac{\kappa t}{m^2}\right)^{1/3}$

Similarity solution:

$$\rho\left(\frac{\kappa t}{m^2}\right)^{1/3} = h\left(\frac{x}{(m\kappa t)^{1/3}}\right) = h(\xi)$$

h ↳ to be found. ξ ↳ $\frac{x}{(m\kappa t)^{1/3}}$

April 30, 2004 Lecture 24

Pick up solution to Hmwk 5

Similarity solution: Ex. nonlinear diffusion ($\nu = \kappa\rho^n$) ($q = -\nu\rho_x$)

$$\begin{cases} \rho_t = \kappa(\rho^n \rho_x)_x \\ \rho \to 0 \text{ "fast" as } |x| \to +\infty \\ \rho(x, t=0) = f(x) \quad \Rightarrow \quad \int_{-\infty}^{+\infty} dx\, \rho(x,t) = m = \text{const} = \int_{-\infty}^{+\infty} f(x)\,dx \end{cases}$$

$\boxed{\text{Method A}}$ $n=1$

Quantity	ρ	x	t	κ	m
Dimension	$[\rho]$	L	T	$\frac{L^2}{T}\frac{1}{[\rho]}$	$[\rho]L$

Non-dimensional parameters $\rho\left(\frac{\kappa t}{m}\right)^{1/3}$ $\xi \equiv \frac{x}{(m\kappa t)^{1/3}}$

Similarity Ansatz: $\rho\left(\frac{\kappa t}{m}\right)^{1/3} = g(\xi)$ g ⟶ to be found

Why?

- before we wrote first a function of 2 parameters and then we took one of them to be small and factored it out as a power;
- here since the equation is non-linear, the unknown function enters in the nondimensional parameters

$u/u_0 = f\left(\frac{a}{\sqrt{\nu t}}, \frac{x}{\sqrt{\nu t}}\right)$ Linear diffusion $u_t = \nu u_{xx}$

Moral: dimensional analysis is powerful tool if you are experienced with the problem and know what to expect

Use stretched coordinates, otherwise.