May 10, 2004 Lecture 27

Pick up: Handouts 19, 10 + paper
Solution to practice Set 5
Solution to Hmwk 6

Pick up Quiz 2 in the end of the class
Place Hmwk in box

Review:

Ex. $\begin{cases} \varepsilon u'' + u' = 1, & 0<x<1 \; ; \; |\varepsilon| \ll 1 \\ u(0) = u(1) = 0 \end{cases}$

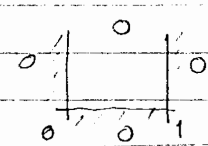
Exact solution:

$$u(x;\varepsilon) = x-1+\frac{e^{-x/\varepsilon}-e^{-1/\varepsilon}}{1-e^{-1/\varepsilon}} \underset{|\varepsilon|\ll 1}{\approx} \underbrace{x-1}_{\alpha(x)} + \underbrace{e^{-x/\varepsilon}}_{\beta(x,\varepsilon)}$$

- $\varepsilon \ll x < 1$: $u(x;\varepsilon) \approx x-1$ OUTER SOLUTION
  $x = O(\varepsilon)$: $u(x;\varepsilon) \approx -1+e^{-\frac{x}{\varepsilon}}$ INNER SOLUTION
  $\Big\}\; \xrightarrow[\frac{x}{\varepsilon}\to\infty]{x\to 0^+} -1$ OVERLAP

$$u(x;\varepsilon) = \underbrace{(\text{OUTER})}_{x-1} + \underbrace{(\text{INNER})}_{-1+e^{-x/\varepsilon}} - \underbrace{(\text{OVERLAP})}_{-1}$$ COMPOSITE FORMULA

Example: $\varepsilon \nabla^2 u - u_x + 2u = 1$



$\varepsilon = 0$ $u(x,y) \equiv h(x,y) = \frac{1}{2}\left[A(y)e^{2x}+1\right]$: CANNOT satisfy all conditions

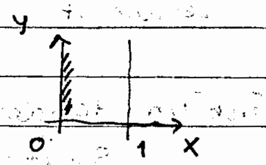
$\Rightarrow$ BOUNDARY LAYERS

! Remark: ε multiplies the highest derivatives in both x & y $\Rightarrow$ we have to look for BL in x & y, all 4 boundaries.

! No need of BL if the highest derivative in x or y is not multiplied by ε

Look for BL:

$\boxed{\text{(i) } x=0:}$ $u(x,y;\varepsilon) = h(x,y) + p(x,y;\varepsilon)$



PDE for p: $\nabla^2 p - \varepsilon^{-1} p_x + \varepsilon^{-1} 2p = -\nabla^2 h$

$p(x,y;\varepsilon) \approx p(\xi, y)$, $\xi = \frac{x}{\varepsilon^\nu}$, $\underrightarrow{\nu > 0}$ to be found

PDE in ξ: $\nabla^2 p = p_{xx} + p_{yy} = \varepsilon^{-2\nu} p_{\xi\xi} + p_{yy}$

PDE is:

$$\varepsilon^{1-2\nu} p_{\xi\xi} + \varepsilon p_{yy} - \varepsilon^{-\nu} p_\xi + 2p = -\varepsilon \nabla^2 h$$

$\downarrow$

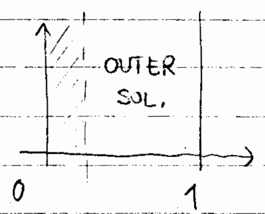
set coeff. in highest derivative to 1

$$p_{\xi\xi} + \varepsilon^{2\nu} p_{yy} - \varepsilon^{\nu-1} p_\xi + 2\varepsilon^{2\nu-1} p = -\varepsilon^{2\nu} \nabla^2 h$$

Considerations:

1) Inside the BL $\xi = O(1)$, $p_{\xi\xi}, p_\xi, p, \nabla^2 h, p_{yy} = O(1)$

2) u inside the BL must reduce to $h(x,y)$ as $\xi \to +\infty$ $\hookrightarrow$ "fast" usually means exponentially



Why exponentially?

$\to$ because of stability reasons: any small perturbation should decay.

$$p_{\xi\xi} + \cancel{\varepsilon^{2\nu} p_{yy}} - \varepsilon^{\nu-1} p_\xi + \cancel{2\varepsilon^{2\nu-1}} p = -\cancel{\varepsilon^{2\nu} \nabla^2} h$$

$$\left[\begin{array}{c} \nu - 1 < 2\nu - 1 \\ \varepsilon^{\nu-1} > \varepsilon^{2\nu-1} \end{array} \right]$$

$\Rightarrow$ $p_{\xi\xi} - \varepsilon^{\nu-1} p_\xi = 0$ $\Rightarrow$ $\boxed{\nu = 1}$ $O(1)$ eqn. for p

! The highest derivative has to be kept in the BL.

solution $\Rightarrow$ $p(\xi, y) = B(y)\, e^{\xi} + C(y)$

not acceptable because grows as $\xi \to +\infty$

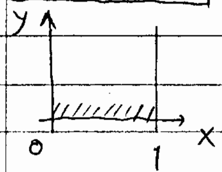
$\Rightarrow$ no BL at $x=0$.

Since there is no BL at $x=0$, $h(x,y)$ has to satisfy the BC

$$u(x=0,y)=0 \Rightarrow A(y)=-1 \quad h(x,y)=\frac{1}{2}(1-e^{2x})$$

$\boxed{\text{(ii)}\quad y=0}$ $\quad u(x,y;\varepsilon)\simeq h(x,y)+w(x,\eta)$

↳ $\eta=\dfrac{y}{\varepsilon^{\mu}}$ $\quad \mu>0$ to be found



PDE for w: $\nabla^2 u = (h_{xx}+w_{xx}) + (\cancel{h_{yy}}+w_{yy}) = -2e^{2x}+w_{xx}+\varepsilon^{-2\mu}w_{\eta\eta}$

PDE in η for w: $\quad -2e^{2x} \quad\quad \varepsilon^{-2\mu}w_{\eta\eta}$

$$\varepsilon(-2e^{2x}+w_{xx}+\varepsilon^{-2\mu}w_{\eta\eta})-(h_x+w_x)+2(h+w)=1$$

satisfied by h

Considerations:

1) Inside BL, $\eta=O(1)$, $w_{\eta\eta}, w_x, h_x, w=O(1)$
2) w as $\eta\to+\infty$ goes to zero exponentially

PDE: $\varepsilon^{1-2\mu}w_{\eta\eta}+\varepsilon w_{xx}-2\varepsilon e^{-2x}-w_x+2w=0$

↳ set this coeff. to 1

$$w_{\eta\eta}+\cancel{\varepsilon^{2\mu}w_{xx}}-\varepsilon^{2\mu-1}w_x+2\varepsilon^{2\mu-1}w=\cancel{2\varepsilon^{2\mu}e^{2x}}$$

we want an order one equation for w $\Rightarrow \mu=\frac{1}{2}$

PDE: $\boxed{w_{\eta\eta}-w_x+2w=0}$ $\quad$! BL does not lead always to ODE's

$$\downarrow$$

$\boxed{(e^{-2x}w)_{\eta\eta}-(e^{-2x}w)_x=0}$ $\quad$ diffusion equation for $v=e^{-2x}w$

Conditions:

- $u(x,y=0)=0 \Rightarrow w(x,\eta=0)=-h(x)=-\frac{1}{2}(1-e^{2x})$
- $w(x,\eta\to+\infty)=0$ so that $u\simeq h$
- since there is no BL at $x=0$ $\quad u(x=0,y)=0 \Rightarrow w(x=0,\eta)=-h(x=0,y)$
  $\Rightarrow w(x=0,\eta)=0$

BL at $y=0$, $\eta = \dfrac{y}{\varepsilon^{1/2}}$; $w = h(x) + W(x,\eta)$

$$\begin{cases} (e^{-2x} W)_{\eta\eta} - (e^{-2x} W)_x = 0 & x>0,\ \eta>0 \\ W(x,\eta=0) = -\frac{1}{2}(1-e^{2x}) & \\ W(x,\eta\to+\infty) = 0 & \\ W(x=0,\eta) = 0 & \end{cases}$$

Handout 10

derivation of the solution

$$\Rightarrow \quad W(x,\eta) = \int_0^x dx' \, \operatorname{erfc}\left(\frac{\eta}{\sqrt{2x'}}\right) e^{2x'}$$

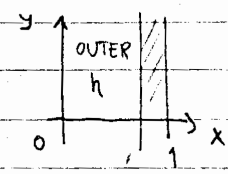
$$\operatorname{erfc}(z) = \frac{2}{\sqrt{\pi}} \int_z^\infty d\lambda \, e^{-\lambda^2}$$

complementary error function

$$w(x,y) = \frac{1}{2}(1-e^{2x}) + W(x,\eta)$$

satisfies PDE to some order, and conditions at $x=0$, $y=0$
but NOT $x=1$.

$\boxed{\text{(iii)} \quad x=1}$



$\sigma = \dfrac{1-x}{\varepsilon^{\alpha}}$, $\alpha>0$ $\quad \sigma = O(1)$

to be found

$$w(x,y;\varepsilon) \approx h(x,y) + W(x,\eta) + V(\sigma,y)$$

PDE for V: (no W because W satisfies the PDE from step 2)

$$\underbrace{\varepsilon^{1-2\alpha}}_{1} V_{\sigma\sigma} + \varepsilon V_{yy} + \varepsilon^{-\alpha} V_\sigma + 2V = 2\varepsilon e^{2x}$$

$$V_{\sigma\sigma} + \cancel{\varepsilon^{2\alpha} V_{yy}} + \varepsilon^{\alpha-1} V_\sigma + \varepsilon^{2\alpha-1} 2V = \cancel{2\varepsilon^{2\alpha} e^{2x}}$$

bigger

$$\Rightarrow \quad V_{\sigma\sigma} + \varepsilon^{\alpha-1} V_\sigma = 0 \qquad \Rightarrow \quad \alpha = 1$$

$$V_{\sigma\sigma} + V_\sigma = 0 \quad \Rightarrow \quad V(\sigma,y) = D(y)\, e^{-\sigma} + E(y)$$

- $V(\sigma, y) \to 0$ as $\sigma \to \infty$ $\Rightarrow$ $E(y) = 0$
- Find $D(y)$

$u(x=1, y) = 0 \Rightarrow h(x=1,y) + w(x=1, \eta) + V(\sigma=0, y) = 0$

$$\Rightarrow \quad D(y) = -\frac{1}{2}(1-e^2) - w\left(1, \frac{y}{\sqrt{\varepsilon}}\right)$$

Combine all pieces together:

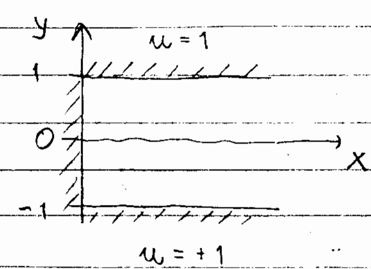
$$\boxed{\begin{aligned} u(x,y;\varepsilon) \approx{}& \frac{1}{2}(1-e^{2x}) + \int_0^x dx'\, e^{2x'} \operatorname{erfc}\left(\frac{y}{\sqrt{2^2\varepsilon x'}}\right) \\ &- \left[\frac{1}{2}(1-e^2) + \int_0^1 dx'\, e^{2x'} \operatorname{erfc}\left(\frac{y}{2\sqrt{\varepsilon x'}}\right)\right] e^{-\frac{1-x}{\varepsilon}} \end{aligned}}$$

beautiful solution because it is not trivial!

This u satisfies the PDE up to order ε (if you admit that highest derivatives are of order 1)
This u satisfies also conditions at $x = 0, 1$, $y = 0$.

Question:
1) condition at $y = \infty$? Satisfied or not?
2) If not why that does not matter?
3) Find a solution that satisfies $y = \infty$?

<u>Example</u>: (Prob. 48)



$$\varepsilon u_{yy} - (1-y^2) u_x + 4y^2 = 0$$

$\uparrow$ BL only in y, $y = \pm 1$

Let $y \to -y$ PDE remains the same, same conditions + symmetric interval

$\Rightarrow$ $u(x,y) = u(x,-y)$ even in y

we can only consider $y \in [-1; 0]$

Set $\varepsilon = 0$: $u_x = \frac{4y^2}{1-y^2} \Rightarrow u = \frac{4xy^2}{1-y^2} + c(y)$

no BL at $x=0$ so $u(x=0,y) = 0 \Rightarrow C(y) = 0$

$h(x,y) \xrightarrow[y \to -1]{} \infty$! not finite value

May 12, 2004 Lecture 28

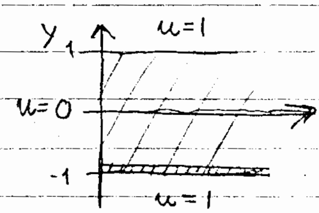
Opt. Redding on Solitons:

- Debnath 9.2-9.6, 11.7, 11.8
- Whitham 13.10-13.12
- Drazin & Johnson, Chs. 1.2

Pick up: Graded Hmwk 5

Boundary layer theory

Example : $\varepsilon u_{yy} - (1-y^2) u_x + 4y^2 = 0$ $u = u(x,y)$



Symmetry : $u(x,y) = u(x,-y)$

I $\varepsilon = 0$:

$u_x = \frac{4y^2}{1-y^2} \Rightarrow u(x,y) = h(x,y) = \frac{4xy^2}{1-y^2} + C(y)$

no boundary layer at $x=0 \Rightarrow h(x=0,y) = 0 \Rightarrow C(y) = 0$

$\Rightarrow h(x,y) = \frac{4xy^2}{1-y^2}$! when $y \to \pm 1$ $h(x,y) \to \infty$

outer solution blows up!

Does not satisfy conditions at $y = \pm 1 \Rightarrow$ boundary layers ?

(i) $y = -1$: $\eta = \frac{1+y}{\varepsilon^\alpha}$, $\alpha > 0$ to be found

$\eta = O(1)$ in the boundary layer : $u(x,y;\varepsilon) = h(x,y) + p(x,\eta)$