NAME:
Pick-up Time:
Return Time:

18.306 Take–Home Quiz # 1
Monday, March 15, 2004

You have 4 hours to do the test.

You are NOT allowed to communicate with anyone, other than the instructor, about this quiz while you are taking it. So, work on the test by yourself!

You can use any books or notes, but you are NOT allowed to use calculators, or any software such as Mathematica or Maple for example, in order to solve the problems.

Read the problems CAREFULLY, especially the given HINT(S). Justify your answers. Try to use all sheets of paper given to you with the test. Cross out what is not meant to be part of your final answer. Total # of points: 30.

DO ALL 3 PROBLEMS!

I. (8 pts) In materials science, the evolution of surfaces of materials is often studied by use of PDE. In the case of evaporation dynamics, where material evaporates to the environment, the surface height $h = h(x, y, t)$ relative to the xy-plane of reference satisfies a complicated PDE. For simplicity, it is assumed that axisymmetry holds, i.e., h depends only on the polar distance $r = \sqrt{x^2 + y^2} \geq 0$ and time $t > 0, h = h(r, t)$; h satisfies the simplified PDE

$$h_t = (A/r)h_r,$$

where A is a positive constant.

Solve this PDE, and discuss why and how any initial smooth, axisymmetric bump $h(r, 0) = H(r)$, where $H(r)$ is monotonically decreasing to 0, finally shrinks to zero as $t \to +\infty$. **Remark:** You are asked to find $h = h(r, t)$ explicitly as function of r and t and discuss implications of the solution.

II.(11 pts) Consider the nonlinear PDE

$$\nu^{-1}\, u_t = u_{xx} + \frac{1-u^2}{u}\,(u_x)^2, \qquad (1)$$

where $u = u(x,t)$, x is a space variable, t is time and ν is a positive constant ($\nu > 0$).

(a) (6 pts) Convert the given PDE (1) to a linear PDE. What is that linear PDE? Show all the steps of derivation in detail. **Hint:** Find a suitable transformation $w = \phi(u)$. You are not asked to solve the PDE for this question (a).

(Continuation of Prob. II.)

(b) (5 pts) Using the result of part (a), solve PDE (1) for $-\infty < x < +\infty$ and $t > 0$ with the initial condition

$$u(x, t = 0) = e^{-x^2}, \quad -\infty < x < +\infty,$$

and the boundary condition $u(x, t) \to 0$ sufficiently fast as $|x| \to \infty$. **Remark:** Beware. You are asked to give $u(x, t)$ explicitly as function of x and t.

III. (11 pts) Consider a function $u = u(x,t)$, defined for $0 \le x \le 1$ and $t \ge 0$, that satisfies the PDE

$$u_t - (x-u)^2 \, u_x = 0. \qquad (2)$$

(a) (7 pts) Suppose that $u(x,t)$ satisfies the given PDE (2) with the conditions

$$u(x,0) = 0, \quad \text{and} \quad u(1,t) = 1.$$

Find and sketch the characteristics for this problem. By using the characteristics, derive an explicit formula for the solution $u(x,t)$ as a function of x and t. Discuss the behavior of $u(x,t)$ at the point $(x,t) = (1,0)$.

(Continuation of Prob. III.)

(b) (4 pts) Now consider a $u(x,t)$ that satisfies the PDE (2), $u_t - (u-x)^2 u_x = 0$, but with the conditions

$$u(x,0) = 0, \quad \text{and} \quad u(1,t) = \frac{1}{2}.$$

Find and sketch the characteristics for this problem. By using the characteristics, derive an explicit formula for the solution $u(x,t)$ as a function of x and t. Discuss the behavior of $u(x,t)$ at the point $(x,t) = (1,0)$.