Hard Problems from Advanced Partial Differential Equations (18.306)

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1. We are given the PDE $\nabla^2\Psi = \Psi_{xx} + \Psi_{yy} = 0$. We must find solutions of the form $\Psi = x^\gamma f(\xi)$, where $\xi \equiv x/y$. We also impose the condition that $\Psi_x - \Psi_y = 0$ along the line $x = y$. What are the possible values for γ and what are the corresponding functions $f(\xi)$ that solve this PDE?

   Solution: Plugging into the PDE, we quickly get that f must fulfill the following ODE:

$$\xi^2(\xi^2+1)f'' + 2\xi(\xi^2+\gamma)f' + \gamma(\gamma-1)f = 0 \tag{1}$$

   We first note that this isn't a standard Sturm-Liuville problem since our undetermined constant γ appears in the f' term. We can convert a second order ODE to a Sturm-Liouville problem only when the undetermined constant, i.e. the eigenvalue, occurs only in the coefficient of the f term. So we must use our ingenuity to tackle this problem from a different perspective.

   We notice that ξ is the cotangent of the angle made by the point (x, y) with the horizontal. This prompts us to try a change of variables so as to take advantage of the angular correspondence to ξ:

$$\xi = \cot\theta \longrightarrow \frac{d}{d\xi} = \frac{d}{d(\cot\theta)} = -\sin^2\theta\frac{d}{d\theta}$$

   Similarly, we also get

$$\frac{d^2}{d\xi^2} = \sin^2\theta\sin 2\theta\frac{d}{d\theta} + \sin^4\theta\frac{d^2}{d\theta^2}$$

   Let's go ahead and define $f(\xi) = g(\cot\theta)$. We will instead solve for g and from there we can get f. Applying our new variable and our new function, we have

$$\begin{aligned}0 &= \xi^2(\xi^2+1)f'' + 2\xi(\xi^2+\gamma)f' + \gamma(\gamma-1)f \\ &= \cot^2\theta\csc^2\theta(\sin^2\theta\sin 2\theta\ g' + \sin^4\theta\ g'') + 2\cot\theta(\cot^2\theta+\gamma)(-\sin^2\theta)\ g' + \gamma(\gamma-1)\ g\end{aligned}$$

$$\implies \quad \cos^2\theta\ g'' - 2\gamma\sin\theta\cos\theta\ g' + \gamma(\gamma-1)\ g = 0. \tag{2}$$

Observing the coefficients of Equation 2, we are reminded of the identity

$$-2\sin\theta\cos\theta = \frac{d}{d\theta}(\cos^2\theta) \tag{3}$$

which helps us notice the following:

$(\cos^{2-\gamma}\theta)\frac{d^2}{d\theta^2}(\cos^\gamma\theta\ g)$

$= \cos^{2-\gamma}[\cos^\gamma\theta\ g'' - 2\gamma\cos^{\gamma-1}\theta\sin\theta\ g' - \gamma(-(\gamma-1)\cos^{\gamma-2}\theta\sin^2\theta + \cos^\gamma\theta)\ g]$

$= \cos^2\theta\ g'' - 2\gamma\cos\theta\sin\theta\ g' + [\gamma(\gamma-1)\sin^2\theta - \gamma\cos^2\theta]\ g$

$= \cos^2\theta\ g'' - 2\gamma\cos\theta\sin\theta\ g' + \gamma(\gamma-1)\ g - \gamma^2\cos^2\theta\ g.$

This looks an awful lot like the left-hand side of Equation 2, so let's use this fact to rewrite the ODE:

$(\cos^{2-\gamma}\theta)\frac{d^2}{d\theta^2}(\cos^\gamma\theta\ g) + \gamma^2\cos^2\theta\ g = \cos^2\theta\ g'' - 2\gamma\sin\theta\cos\theta\ g' + \gamma(\gamma-1)\ g = 0.$

This yields

$$\frac{d^2}{d\theta^2}(\cos^\gamma\theta\ g) = -\gamma^2(\cos^\gamma\theta\ g). \tag{4}$$

Now, we rejoice, because letting $\phi = \cos^\gamma\theta\ g$, produces a second-order linear ODE with constant coefficients:

$$\phi'' = -\gamma^2\ \phi \implies \phi(\theta) = A\cos(\gamma\theta) + B\sin(\gamma\theta)$$
$$\implies g(\theta) = (\cos^{-\gamma}\theta)[A\cos(\gamma\theta) + B\sin(\gamma\theta)]$$

is the general solution for g.

Now, let us consider the given conditions. $\Psi \to 0$ as $y \to \infty$ for constant x means that $\Psi(x, y \to \infty) = x^\gamma f(\xi)|_{\xi\to 0} = x^\gamma g(\text{arccot}(\xi))|_{\xi\to 0} = 0$. Thus we can deduce that $g(\text{arccot}(0)) = g(\pi/2) = 0$ whenever $x \neq 0$. Looking back to our general solution, we see that when $\theta = \pi/2$, the prefactor $(\cos^{-\gamma}\theta)$ becomes $0^{-\gamma}$ which will go to infinity and violate our condition unless $\gamma < 0$. Thus we require $\gamma < 0$. But under this requirement, $g(\pi/2)$ automatically equals 0, meaning that $\gamma < 0$ is the only constraint needed in the $x \neq 0$ case. Now, we must check if we need to impose anymore restrictions on γ so as to ensure that our condition holds when $x = 0$. In this case, we need the leading term x^γ to cancel in order to prevent Ψ from going to ∞ along the y-axis. Notice that $\cos^{-\gamma}\theta = \cos^{-\gamma}(\operatorname{arccot}(x/y)) = (\sqrt{x^2+y^2}/x)^\gamma$. Thus, we re-write $\Psi(x,y) = \sqrt{x^2+y^2}\ [A\cos(\gamma * \operatorname{arccot}(x/y)) + B\sin(\gamma * \operatorname{arccot}(x/y))]$, and see that when $x = 0$, Ψ is a constant times y^γ. Ψ goes to zero when $y \to \infty$ as long as γ is negative, thus we need not add anymore restrictions to γ. So, we have that the only constraint is $\gamma < 0$.

For the other condition, we start by writing it using $\Psi = x^\gamma g(\theta)$.

$$\Psi_x = \gamma x^{\gamma-1} g + x^\gamma \theta_x = x^{\gamma-1}(\gamma g - \sin\theta\cos\theta g')$$

$$\Psi_y = x^{\gamma-1} g' \theta_y = x^{\gamma-1} g' \cos^2\theta$$

$$\therefore \Psi_x - \Psi_y = x^{\gamma-1}(\gamma g - (\sin\theta\cos\theta + \cos^2\theta)g').$$

Our condition says this needs to be zero whenever $x = y$, or, equivalently, $\theta = \pi/4, 5\pi/4$. Thus in the $\theta = \pi/4$ case, we have

$$\gamma g(\pi/4) - g'(\pi/4) = 0 \implies \gamma g(\pi/4) = g'(\pi/4).$$

Plugging in our g:

$$g'(\pi/4) = (\gamma\cos^{-\gamma-1}\theta\ \sin\theta(A\cos\gamma\theta + B\sin\gamma\theta) + \gamma\cos^{-\gamma}\theta(-A\sin\gamma\theta + B\cos\gamma\theta))|_{\theta=\pi/4}$$

$$= (\gamma\cos^{-\gamma}\frac{\pi}{4})[A\cos\gamma\frac{\pi}{4} + B\sin\gamma\frac{\pi}{4} - A\sin\gamma\frac{\pi}{4} + B\cos\gamma\frac{\pi}{4}].$$

$\therefore$ The condition $\gamma g(\pi/4) = g'(\pi/4)$ implies:

$$A\cos(\gamma\pi/4) + B\sin(\gamma\pi/4) = A\cos(\gamma\pi/4) + B\sin(\gamma\pi/4) - A\sin(\gamma\pi/4) + B\cos(\gamma\pi/4)$$

$$\implies B\cos(\gamma\pi/4) = A\sin(\gamma\pi/4).$$

This yields $B = A\tan(\gamma\pi/4)$ when γ is not even, and $A = 0$ with B arbitrary when γ is even. In the case of $\theta = 5\pi/4$, we easily reach the analagous result

$$B\cos(5\gamma\pi/4) = A\sin(5\gamma\pi/4)$$

into which we plug in our recently determined expressions for A and B, yielding

$$\tan(\gamma\pi/4) = \tan(5\gamma\pi/4).$$

This combined with the π-periodicity of the tangent function means that $\gamma\pi/4 = 5\gamma\pi/4 + \pi n$ for some integer n. Therefore, we deduce the additional constraint that γ must be an integer. So, altogether, we have:

$$\boxed{\text{For } \gamma \in \mathbb{Z} \text{ and } \gamma < 0, \quad \Psi = \begin{cases} A * x^{\gamma} \cos^{-\gamma}\theta \; [\cos(\gamma\theta) + \tan(\gamma\pi/4)\sin(\gamma\theta)] & \text{if } \gamma \text{ is not even,} \\ B * x^{\gamma} \cos^{-\gamma}\theta \; [\sin(\gamma\theta)] & \text{if } \gamma \text{ is even.} \end{cases}}$$

where $\theta = \operatorname{arccot}(\xi)$ as previously defined, and the constants A and B are arbitrary.

2. The nonlinear KdV equation $u_t - 6uu_x + u_{xxx} = 0$ can be shown to have a similarity solution of the form $u = -(3t)^{-2/3}g(\eta)$ for $\eta = x(3t)^{-1/3}$. Applying this solution reduces the PDE to the ODE $g'''(\eta) + (6g(\eta) - \eta)g'(\eta) - 2g(\eta) = 0$. Show that this ODE reduces to the second-order ODE $V'' - \eta V - 2V^3 = 0$ upon a proper choice of the constant μ in the variable change $g(\eta) = \mu(dV(\eta)/d\eta) - V(\eta)^2$ (and given that V decays exponentially for large η). How does the solution relate to Airy functions for large η?

   Solution: We first plug the given variable change directly into the ODE for g, yielding the following ODE for V:

$$\mu V'''' - 2VV''' + 6\mu^2 V'V'' - 6V'V'' - 6\mu V^2 V'' - \mu\eta V'' -$$

$$12\mu V(V')^2 + 12V^3V' + 2\eta VV' - 2\mu V' + 2V^2 = 0.$$

   We try $\mu = 1$. A keen observation allows us to notice that with this selection for μ, the ODE can be written as

$$\frac{d^2}{d\eta^2}(V'' - \eta V - 2V^3) - 2V\frac{d}{d\eta}(V'' - \eta V - 2V^3) = 0.$$

   Letting $F(\eta) = \frac{d}{d\eta}(V'' - \eta V - 2V^3)$, we can rewrite the ODE as

$$\frac{dF}{d\eta} = 2VF$$

   which quickly reduces to

$$\frac{d}{d\eta}(\log F) = 2V$$

leaving us with the general solution

$$F(\eta) = Ae^{2\int V(\eta)d\eta}$$

for some constant A. We are given that V is exponentially decaying for large η, so in this regime, the antiderivative $\int V d\eta$ is also an exponentially decaying function. And thus as η gets large, the right side approaches $Ae^{2*0} = A$. So as $\eta \to \infty$,

$$F(\eta) = \frac{d}{d\eta}(V'' - \eta V - 2V^3) \to A$$

$$\implies \quad V'' - \eta V - 2V^3 \to A\eta + B \tag{5}$$

for some constant B. As $\eta \to \infty$, all three terms on the left side of (5) approach 0 since V is exponentially decaying. Therefore, A and B must both be 0, and our ODE has succesfully reduced to the second-order ODE

$$V'' - \eta V - 2V^3 = 0 \tag{6}$$

This marks the completion of the first part of this problem.

V decaying exponentially means that for large η, the V^3 term in Equation 6 is exponentially smaller than the other two terms. Likewise we will neglect the V^3 term. We are left to analyze the ODE

$$V'' - \eta V = 0 \tag{7}$$

for $\eta \gg 1$. We note that if we could somehow convert every $(d/d\eta)$ into η and vice versa, then the ODE would be first-order and easily solvable. Recalling that Fourier Transforms have this feature, we express V in terms of its transform:

$$V(\eta) = (1/2\pi)\int_{-\infty}^{\infty} \hat{V}(k)\ e^{ik\eta}dk.$$

Now, we can write

$$V''(\eta) = (1/2\pi)\int_{-\infty}^{\infty} -k^2\ \hat{V}(k)\ e^{ik\eta}dk$$

and

$$\eta V(\eta) = \eta\int_{-\infty}^{\infty} \hat{V}(k)\, e^{ik\eta}dk = \int_{-\infty}^{\infty} \hat{V}(k)\,\eta\, e^{ik\eta}\,dk = \hat{V}(k)\ \frac{e^{ik\eta}}{i}\Bigg|_{k=-\infty}^{k=\infty} - \int_{-\infty}^{\infty} \hat{V}'(k)\ \frac{e^{ik\eta}}{i}\ dk.$$

As long as the integration contour is chosen to ensure $\hat{V}(k)e^{ik\eta}\Big|_{k\to-\infty}^{k\to+\infty} = 0$, Equation 7 can be written in terms of $\hat{V}$ as:

$$(1/2\pi)\int_{-\infty}^{\infty}\left(-k^2\hat{V}(k) + \frac{\hat{V}'(k)}{i}\right)\ e^{ik\eta}dk = 0$$

By uniqueness of the Fourier Transform, this means

$$\frac{\hat{V}'(k)}{i} - k^2\hat{V}(k) = 0. \tag{8}$$

This first order ODE has the solution

$$\hat{V}(k) = Ce^{ik^3/3}. \tag{9}$$

Now that we have a candidate for $\hat{V}(k)$, we can only accept it once we verify that there is an appropriate contour whose endpoints are at $\pm\infty$ such that $f(k) = \hat{V}(k)e^{ik\eta}\Big|_{k\to-\infty}^{k\to+\infty} = 0$. The function $f(k) = Ce^{i(k\eta+k^3/3)}$ is analytic $\forall k \in \mathcal{C}$, so we can deform the integration contour however we wish within $\mathcal{C}$ as long as the endpoints remain at $\pm\infty$. Let $k = re^{i\theta}$. The exponent in $f(k)$ is dominated by its cubic term for $|k|$ large, thus for the sake of detecting decay behavior, we may neglect the $k\eta$ term. The real part of this reduced exponent is $Re(ik^3/3) = Re(ir^3e^{3i\theta}/3) = -(r^3/3)\sin(3\theta)$. Decay behavior requires this to be negative in the region through which the ends of the contour pass. Consequently, the ends must approach $\pm\infty$ through regions fulfilling $2\pi n < 3\theta < \pi + 2\pi n$. This "good" region can be more simply written as $\begin{cases} 0 < \theta < \pi/3 \\ 2\pi/3 < \theta < \pi \\ 4\pi/3 < \theta < 5\pi/3. \end{cases}$

So, our goal is met for any contour that approaches $+\infty$ through the $0 < \theta < \pi/3$ region, $-\infty$ through the $2\pi/3 < \theta < \pi$ region, and does not attain a large magnitude in a region outside the three listed above. For example, the contour $k(t) = t + \frac{i}{1+t^2}$ would suffice. Therefore, we can finally say that the $\hat{V}$ we received is indeed legitimate.

Transforming out, our solution for V is

$$\begin{aligned} V(\eta) &= (1/2\pi)\int_{-\infty}^{\infty} Ce^{i(k\eta+k^3/3)}dk && (10) \\ &= (1/\pi)\int_{0}^{\infty} C\cos(k\eta + k^3/3)\ dk && (11) \\ &= \boxed{C * Ai(\eta)} && (12) \end{aligned}$$

This solution, though only one of two independent solutions, is the one we desire since it decays exponentially as $\eta \to +\infty$.