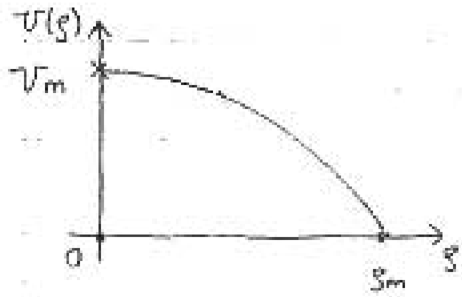
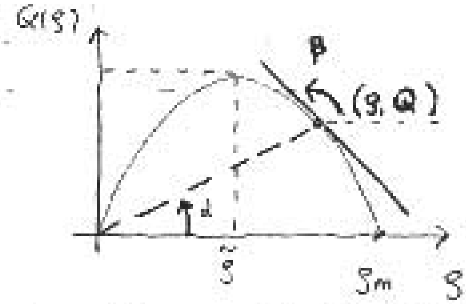
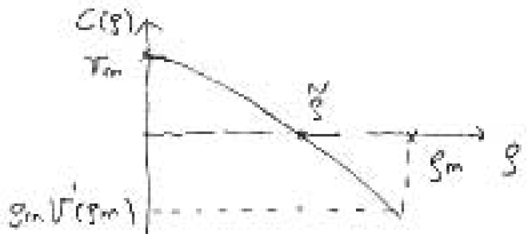




↖ maximum value of ρ $\qquad tg\,\alpha = \frac{Q}{\rho} = V$

since V goes down, the slope has to go down.

$c(\rho) = Q'(\rho)$: propagation speed
i.e. this is the slope $tg\,\beta$



$\Rightarrow c'(\rho) < 0$ $\quad c(\rho)$ decreases
as the density increases, the drivers tend to decrease their speed

PDE for ρ : $\frac{\partial \rho}{\partial t} + \underbrace{Q'(\rho)}_{c(\rho)} \frac{\partial \rho}{\partial x} = 0$ $\qquad c'(\rho) < 0$

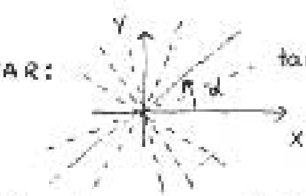
February 18, 2004 Lecture 5

Review Session: FRI 4-5pm
Pick up Handout 4

<u>Review Lecture 4</u> :

Eikonal equ. $\begin{cases} S_x^2 + S_y^2 - 1 = 0 \\ S(0,y) = u_0(y), \; y = s \end{cases}$ $\qquad (q_0(s) = u_0'(s), \; q_0^2 + p_0^2 = 1)$

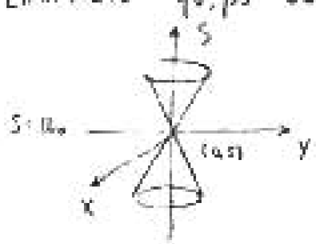
CHAR:



$\tan\alpha = \frac{q_0(s)}{p_0(s)}$ $\qquad q_0(s)x - p_0(s)y = K_s$

(for each point $(0,s)$ a line passes through the slope is done by q_0/p_0)

Eliminate q_0, p_0 but keep s fixed $\Rightarrow$ consider all possible slope of CHAR

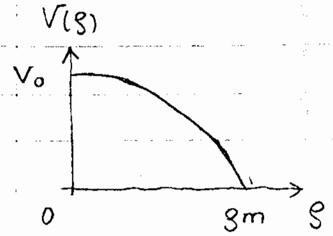


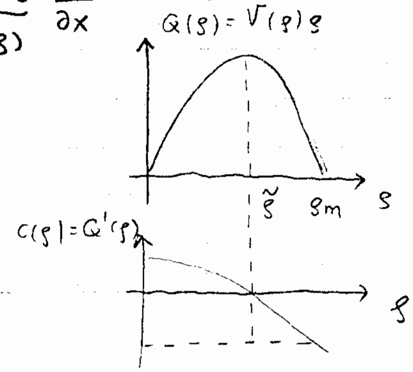
Monge cone

(all points lying on the disk are affected by the data $(0,s)$, all r)

(all ways the CHAR can affect the solution by the data at point $(0,s)$ → 3D cone)

Traffic flow $\quad \dfrac{\partial \rho}{\partial t} + \underbrace{Q'(\rho)}_{c(\rho)} \dfrac{\partial \rho}{\partial x} = 0$





Solve the IVP:

$$\begin{cases} \rho_t + c(\rho)\rho_x = 0 & -\infty < x < \infty \\ \rho(x,0) = f(x) \text{ known} & t > 0 \end{cases}$$

Along CHAR: $\quad \dfrac{dt}{1} = \dfrac{dx}{c(\rho)} = \dfrac{d\rho}{0} \quad \Rightarrow$ ① $\rho = K_1 = \text{const}$ along CHAR
(quasi-linear) $\qquad$ ② $x = c(\rho)t + K_2$ (CHAR-lines)

✓ the slope of the CHAR depends on the solution

Initial data:

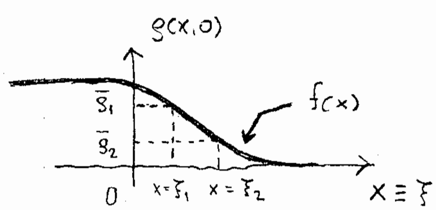
$x = \xi,\ t = 0: \quad \rho = f(\xi) \quad \Rightarrow$ ① $K_1 = f(\xi)$

② $\xi = 0 + K_2 \quad \Rightarrow \quad K_2 = \xi$

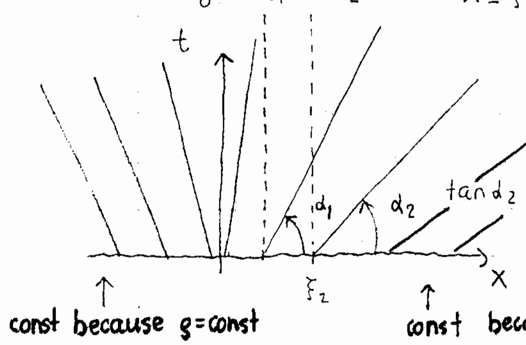
$$\Rightarrow \begin{cases} \rho = f(\xi) \\ x = \underbrace{c(f(\xi))}_{\substack{\text{composite function} \\ F(\xi)}}\, t + \xi \end{cases}$$

<u>Scenario 1</u>: $\quad F'(\xi) \geq 0 \quad$ in particular $\begin{cases} c'(\rho) < 0 \\ f'(\xi) \leq 0 \end{cases}$ $\quad F'(\xi) = c'(\rho) f'(\xi) \geq 0$ ($f(\xi) \Leftrightarrow \rho$)

↗ consistent choice with traffic flow



(that's the variable we choose to parametrize the data)



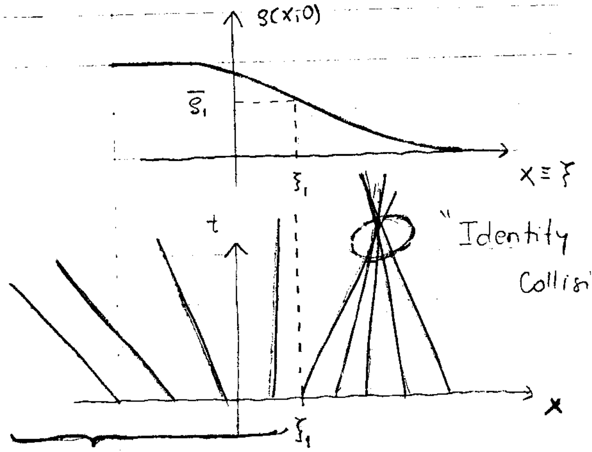
$\tan\alpha_1 = \dfrac{1}{F(\xi_1)} > \tan\alpha_2 = \dfrac{1}{F(\xi_2)}$ because $F'(\xi) \geq 0$

$\tan\alpha_2 = \dfrac{1}{F(\xi_2)} = \dfrac{1}{c(f(\xi_2))}$

What's the solution on the CHAR emanating from ξ_2

$\rho = \bar{\rho}_2 = f(\xi_2)$

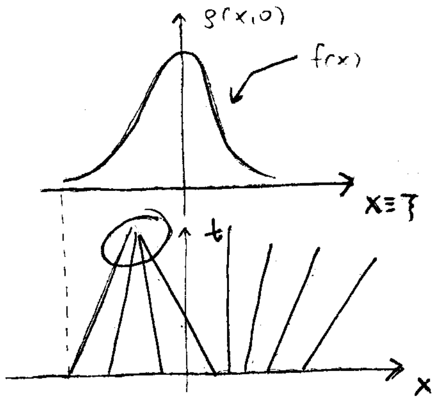
Scenario 2 $F'(\xi) \leq 0$ in particular $\begin{cases} c'(\rho) > 0 \\ f'(\xi) \leq 0 \end{cases}$, $\rho < \bar{\rho}_1$



same as before

unphysical scenario

We can obtain the same situation if $\begin{cases} c'(\rho) < 0 \\ f'(\xi) \geq 0 \end{cases}$



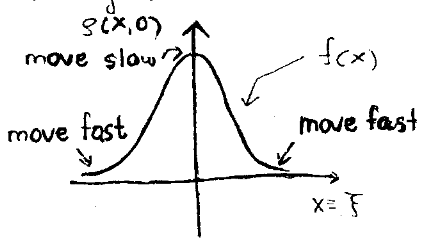
The problem is that we have a density that is multidefined. The condition to happen is

$$\boxed{F'(\xi) \leq 0}$$

ρ is multi-valued

What goes wrong with our model?

$c = c(\rho)$
↑
only



⟶ Remedy: allow car speed to depend on ρ, ρ_x

$$q = Q(\rho) - V\rho_x$$

(this leads to Burger's eq.)

Systematic treatment:

$$\begin{cases} \rho_t + c(\rho)\,\rho_x = 0 \\ \rho(x,0) = f(x) \end{cases} \quad -\infty < x < \infty\,,\ t > 0$$

Define: $F(\xi) = c(f(\xi))$
$F'(\xi) < 0$

Correspondence that shows the influence of the data on the solution:

$$(x, t) \curvearrowright \xi(x,t)$$

- $\rho = f(\xi) \Rightarrow \begin{cases} \rho_x = f'(\xi)\,\xi_x \\ \rho_t = f'(\xi)\,\xi_t \end{cases}$ (A)

- along the CHAR $x = F(\xi)t + \xi \Rightarrow \begin{cases} 1 = F'(\xi)\xi_x t + \xi_x \\ 0 = F'(\xi)\xi_t t + F(\xi) + \xi_t \end{cases}$

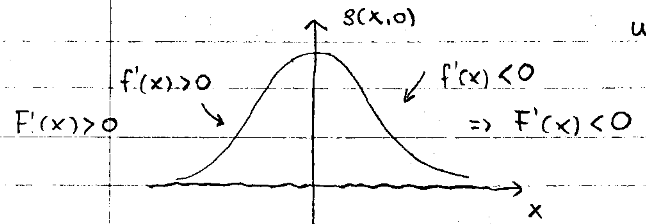
$\Rightarrow \xi_x = [1 + F'(\xi)t]^{-1}, \quad \xi_t = -F(\xi)[1 + tF'(\xi)]^{-1}$ (B)

From equs (A) and (B): $\rho_x = \dfrac{f'(\xi)}{1+F'(\xi)t}, \quad \rho_t = \dfrac{-f'(\xi)F(\xi)}{1+F'(\xi)t}$

Is it possible that $\rho_x, \rho_t \to \infty$?

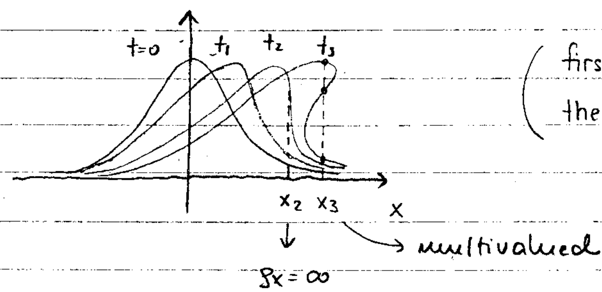
$1 + tF'(\xi) = 0 \Rightarrow t = -\dfrac{1}{F'(\xi)}$ possible if $F'(\xi) < 0$ (since $t > 0$)

What happens to ρ?



we take $c'(\rho) > 0$

$\Rightarrow$ high density propagates faster



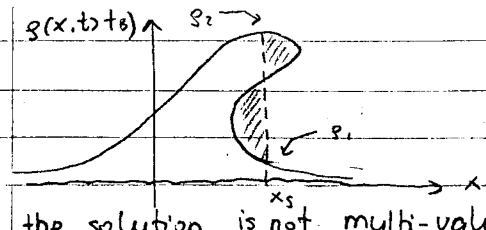
(first-order PDE gives this
the remedy is 2-order PDE, shock)

What is the earliest time for all possible ξ that this happens?

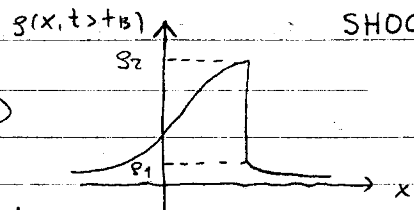
$$t_B = \min_{\xi} \frac{1}{|F'(\xi)|} = \frac{1}{|F'_{\xi}|_{max}} \quad \text{breaking time}$$

Remedies:

A.) Allow for a discontinuous ρ:



$\Rightarrow$

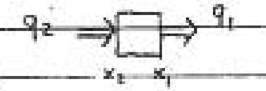


SHOCK WAVE

the solution is not multi-valued
but rather changes abruptly from ρ_2 to ρ_1

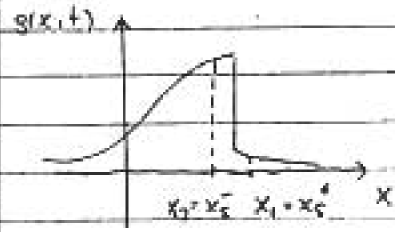
Where is the "jump"? $x_s = x_s(t)$

Recall: conservation law (integration form) $\frac{d}{dt}\int_{x_2}^{x_1} g(x,t)\,dx = q(x_2,t) - q(x_1,t)$



in order to obtain the PDE, we assumed that $g \in C^1$ (differentiable)

Solutions to the integration form are called WEAK solutions of PDE. The shockwave is a weak solution - it doesn't satisfy the PDE at all points.



- $\int_{x_2}^{x_1} \cdots = \int_{x_2}^{x_s(t)} \cdots + \int_{x_s(t)}^{x_1} \cdots$
- Recall: $\frac{d}{dt}\int_{a(t)}^{b(t)} dx\, g(x,t) = b'(t) g(b,t) - a'(t) g(a,t) + \int_{a(t)}^{b(t)} dx \frac{\partial g}{\partial t}(x,t)$

Conservation law:

$$x_s'(t)\, g(x_s^-,t) - x_s'(t)\, g(x_s^+,t) + \int_{x_1}^{x_2} dx \frac{\partial g}{\partial t} = q(x_2,t) - q(x_1,t)$$

take $x_2 \to x_1$

$$x_s'(t) = \frac{q(x_2,t) - q(x_1,t)}{g(x_2,t) - g(x_1,t)} = \frac{q(x_s^-,t) - q(x_s^+,t)}{g(x_s^-,t) - g(x_s^+,t)} \Rightarrow U = \frac{q_2 - q_1}{g_2 - g_1}$$

values right and left of the shock

February 23, 2004 Lecture 6

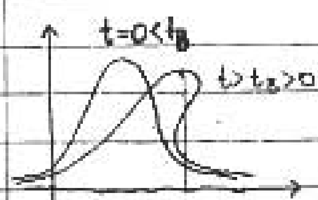
Reading Whitham 5.1-5.4

Review session on Transforms: FRI 4-5pm

Review IVP $\begin{cases} g_t + c(g) g_x = 0 \quad (c(g) = Q'(g)) \\ g(x,0) = f(x);\ x = \xi \end{cases}$

$F(\xi) = c(f(\xi))$

↳ assume $F'(\xi) < 0$ somewhere

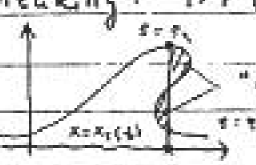


wave breaks for $t > t_B$, $t_B = 1/|F'(\xi)|_{max}$

Condition for breaking: $1 + F'(\xi)t = 0$

the pos cannot be arbitrary

Remedies: (A) Place "shock":



Restriction: "shock speed" $\frac{dx_s}{dt} = \frac{Q(g_1) - Q(g_2)}{g_1 - g_2}$