18.306 : Advanced Partial Differential Equations with Applications

Handout 3 : (Brief) Review of Ordinary Differential Equations (ODE)

[OPTIONAL READING]

by D. Margetis

In this note, I review briefly basic solvable ODE by following the presentation in Chap. 1 of Hildebrand's "Advanced calculus for applications." The specific details of each method do not matter **so much**; you may already know how to find solutions by alternative routes. What matters is to "have a feeling" of what a right method might be, i.e., to recognize an exactly solvable ODE and then try to apply any reasonable method. The list of cases or methods given here is by no means exhaustive.

Case 1 (trivial) :

$$u' \equiv \frac{du}{dx} = h(x) \quad : \text{known} . \qquad (1)$$

$$\overset{\text{Integrate}}{\Longrightarrow} \quad u(x) = \int h(x)\, dx + C_1 , \qquad C_1 : \text{arbitrary constant.} \qquad (2)$$

Case 2 : First-order ODE in separable form, viz.,

$$F(x)\, G(u) + f(x)\, g(u)\, u' = 0 , \qquad F, G, f, g : \text{known}. \qquad (3)$$

$$\overset{\text{Solve}}{\Longrightarrow} \quad \int \frac{F(x)}{f(x)}\, dx + \int \frac{g(u)}{G(u)}\, du = C_1 = \text{const.} \qquad (4)$$

Equation (4) gives a relation between u and x by which u is determined explicitly or implicitly. (We may not always be able to carry out the integrations.) The governing equation (3) may of course be nonlinear (in u).

Case 3 : Non-homogeneous $(h(x) \neq 0)$, linear 1st-order ODE, viz.,

$$u'(x) + a_1(x) \cdot u(x) = h(x) \quad , \qquad a_1(x),\ h(x) : \text{given}. \tag{5}$$

A systematic (rather formal) way to solve this ODE follows. Define an integrating factor $p = p(x)$ such that (5) becomes

$$\frac{d}{dx}(pu) = ph . \tag{6}$$

Equations (5) and (6) are equivalent if $p(x)$ satisfies the ODE

$$\frac{1}{p} \frac{dp}{dx} = a_1(x) \quad , \tag{7}$$

which is of the form (3). Hence,

$$p(x) = \exp\left[\int dx\ a_1(x) \right]. \tag{8}$$

It follows from (6) that

$$u(x) = \frac{1}{p(x)} \cdot \left[\int dx\ p(x) h(x) + C_1 \right] \quad , \qquad C_1 = \text{const.}, \tag{9}$$

where p is given by Eq. (8).

Case 4 : Linear, nth-order ODE with constant coefficients, viz.,

$$u^{(n)}(x) + a_{n-1} u^{(n-1)}(x) + a_{n-2} u^{(n-2)}(x) + \cdots + a_1 u'(x) + a_0 u(x) = g(x) \quad , \tag{10}$$

where a_i are given constants $(i = 0, 1, \ldots, n)$ and $g(x)$ is a given function of x ($u^{(k)}$ denotes the kth derivative of $u = u(x)$).

General solution of (10):

$$u = u_h + u_p \quad , \tag{11}$$

where u_h is the general solution to the homogeneous equation $(g \equiv 0)$ and u_p is a particular solution to (10).

Steps : ① Find u_h by setting $g \equiv 0$ and $u = u_h = e^{\lambda x}$. Hence, solve

$$\lambda^n + a_{n-1}\lambda^{n-1} + a_{n-2}\lambda^{n-2} + \dots + a_1\lambda + a_0 = 0. \tag{12}$$

- If (12) has n distinct roots $\{\lambda_k\}$ then ($\lambda_i \neq \lambda_j$ for $i \neq j$)

$$u_h(x) = \sum_{k=1}^{n} C_k e^{\lambda_k x} , \qquad C_k = \text{const}. \tag{13}$$

- If $\lambda_1 = \lambda_2 = \dots = \lambda_m$ (root of multiplicity m) then replace

$$\sum_{k=1}^{m} C_k e^{\lambda_k x} \longrightarrow e^{\lambda_1 x}(c_1 + c_2 x + \dots + c_m x^{m-1}). \tag{14}$$

Example 1: Solve $u^{(5)} - 3u^{(4)} + 3u''' - u'' = 0.$

The polynomial $\lambda^5 - 3\lambda^4 + 3\lambda^3 - \lambda^2 = \lambda^2(\lambda^3 - 3\lambda^2 + 3\lambda - 1) = \lambda^2(\lambda - 1)^3$

has roots $\lambda = 0$ (double) and $\lambda = 1$ (triple). Hence,

$$u(x) = e^{\mathbf{1}x}(c_1 + c_2 x + c_3 x^2) + e^{0\cdot x}(c_4 + c_5 x) = e^x(c_1 + c_2 x + c_3 x^2) + c_4 + c_5 x. \tag{15}$$

Ⓘ Find the particular solution u_p of (10) by one of the following methods:

(i) Method of undetermined coefficients.

This method works only if the RHS of (10) is $g(x) = x^m$ (m: integer), $\sin(qx)$, $\cos(qx)$, e^{px}, and/or sums and/or products of two or more such functions. Notice that each of these functions, or their products, has only a finite number of independent derivatives. We define as "family" of a function $f(x)$ the set of linearly independent functions of which $f(x)$ and its derivatives in x are linear combinations (see Table I).

Function $g(x)$	"Family"
x^m, m: integer	$x^m, x^{m-1}, \dots, x^2, x, 1$
$\sin(ax)$	$\sin(ax)$, $\cos(ax)$
$\cos(ax)$	$\sin(ax)$, $\cos(ax)$
e^{bx}	e^{bx}

Table I

Accordingly, the family of a function that is the product of n such terms (left column) consists of all possible products of n factors, in which one factor in each product is from the family (right column) of the corresponding term.

Example 2: Find the family of $g(x) = \underset{\text{(multiply)}}{x^3 \cdot \sin x}$

Family of x^3 : $\{x^3, x^2, x, 1\}$; Family of $\sin x$: $\{\sin x, \cos x\}$.

Family of $x^3 \sin x$: $\{x^3 \sin x, x^2 \sin x, x \sin x, \sin x, x^3 \cos x, x^2 \cos x, x \cos x, \cos x\}$.

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To proceed, identify $g(x)$ in the RHS of (10) as a linear combination of terms such as those in left column of Table I, or their products. **Then,** construct the family of each term. If any family has a member that is a solution of the homogeneous equation (with $g \equiv 0$), then multiply each member of that family by the lowest integral power of x for which no member of the new family is a solution of the homogeneous equation.

A particular solution of (10) is a linear combination of all members of resulting families.

The coefficients are determined by substitution in (10)

Example 3: Solve $u''' - u' = 2x + 1 - 4\cos x + 2e^{x}$ (16)

Homogeneous eqn (I): $u''' - u' = 0 \quad \longrightarrow \quad u_h(x) = c_1 \cdot 1 + c_2 e^{x} + c_3 e^{-x}$ $(\lambda = 0, 1, -1)$ (17)

Particular soln (II): Set $g(x) = 2x + 1 - 4\cos x + 2e^{x}$

Families: $\{x, 1\}$, $\{1\}$, $\{\cos x, \sin x\}$, $\{e^{x}\}$.

The second family is discarded because it is contained in the first one.

So, families are $\{x, 1\}$, $\{\cos x, \sin x\}$, $\{e^{x}\}$.

The 1 in the first family is a solution to the homogeneous eqn (for $c_2 = c_3 = 0$).

Hence, $\{x, 1\} \xrightarrow{\text{Replaced}} \{x^2, x\}$.

The e^{x} in the third family is a solution to the homogeneous eqn (for $c_1 = c_3 = 0$)

Hence, $\{e^{x}\} \to \{xe^{x}\}$.

New families: $\{x^2, x\}$, $\{\cos x, \sin x\}$, $\{xe^{x}\}$

We seek a particular solution as a linear combination of the members $x, x^2, \cos x, \sin x, xe^{x}$:

$$u_p = Ax^2 + Bx + C\cos x + D\sin x + E \cdot xe^{x}; \qquad (18)$$

A, B, C, D, E: to be determined.

By substitution in (16),

$$A = -1, \quad B = -1, \quad C = 0, \quad D = 2, \quad E = 1. \qquad (19)$$

General solution of (16):

$$u = u_h + u_p = c_1 + c_2 e^{x} + c_3 e^{-x} - x^2 - x + 2\sin x + xe^{x}. \qquad (20)$$

Alternatively, you may proceed along the lines of using $D \equiv \frac{d}{dx}$ from Prof. Cheng's notes for 18.305.

(ii) Method of variation of parameters

This is an alternative method for finding a particular solution $u_p(x)$ to Eq. (10) provided that we know n linearly independent solutions $u_k(x)$ $(k=1,2,\dots,n)$ of the homogeneous equation, i.e., $u_h(x)=\sum_{k=1}^{n} c_k u_k(x)$. This method also works for linear ODE with coefficients that are functions of x (see Case 5 below).

We assume that

$$u_p(x)=\sum_{K=1}^{n} C_K(x)\,u_K(x) \quad , \qquad \begin{array}{l} u_K:\text{ known,} \\ \text{by } g\equiv 0 \end{array} \qquad (21)$$

where $C_k(x)$ are to be determined.

$$\text{Differentiate}\Longrightarrow \quad u_p'(x)=\sum_{K=1}^{n} C_K u_K' + \sum_{K=1}^{n} C_K' u_K \,. \qquad (22)$$

To get an equation for C_K, we require that

$$\sum_{K=1}^{n} C_K' u_K = 0 \,. \qquad (23)$$

It follows that

$$u_p''(x)=\sum_{K=1}^{n} C_K' u_K' + \sum_{K=1}^{n} C_K u_K'' \,. \qquad (24)$$

We further require that the terms involving derivatives of C_K vanish:

$$\sum_{K=1}^{n} C_K' u_K' = 0. \qquad (25)$$

By repeating this procedure $n-1$ times, we impose

$$\sum_{K=1}^{n} C_K' u_K^{(m)} = 0 \quad , \qquad m=0,1,\dots,\ n-2. \qquad (26)$$

Finally, because of (26), we find

$$u_p^{(m)}(x)=\sum_{K=1}^{n} C_K u_K^{(m)}, \quad m=1,2,\dots,n-1, \qquad (27)$$

and

$$u_p^{(n)}(x)=\sum_{K=1}^{n} C_K u_K^{(n)} + \sum_{K=1}^{n} C_K' u_K^{(n-1)}. \qquad (28)$$

By substitution of Eqs. (27) and (28) in (10) we obtain

$$\sum_{k=1}^{n} C_k \left[\cancel{u_k^{(n)} + a_{n-1} u_k^{(n-1)}} + \dots + a_1 u_k' + a_0 u_k \right] + \sum_{k=1}^{n} C_k' \, u_k^{(n-1)} = g(x). \qquad (29)$$

The first term in the LHS <u>vanishes</u> because $u_k(x)$ are solutions of the homogeneous equation (by definition of u_k). Hence,

$$\sum_{k=1}^{n} C_k' \, u_k^{(n-1)} = g(x). \qquad (30)$$

Equations (26) and (30) form a system of n equations for the n unknown functions $C_k(x)$:

$$\left\{ \begin{array}{l} C_1'(x)\, u_1(x) + C_2'(x)\, u_2(x) + \dots + C_n'(x)\, u_n(x) = 0 \\ C_1'(x)\, u_1'(x) + C_2'(x)\, u_2'(x) + \dots + C_n'(x)\, u_n'(x) = 0 \\ \quad \vdots \\ C_1'(x)\, u_1^{(n-1)}(x) + C_2'(x)\, u_2^{(n-1)}(x) + \dots + C_n'(x)\, u_n^{(n-1)}(x) = g(x). \end{array} \right. \qquad (31)$$

We solve this system by applying Cramers rule; because of the independence of $u_k(x)$, the system always has a solution.

<u>Example 4</u>: I solve the ODE of Ex. 3 by the method of variation of parameters

$$u_p(x) = C_1(x) + C_2(x)\, e^x + C_3(x)\, e^{-x}. \qquad (u_1(x) = 1,\ u_2(x) = e^x,\ u_3(x) = e^{-x}) \ (32)$$

Equations for C_1, C_2, C_3:

$$\left\{ \begin{array}{l} C_1' + C_2'\, e^x + C_3'\, e^{-x} = 0 \\ C_1' \cdot 0 + C_2' \cdot e^x - C_3'\, e^{-x} = 0 \\ C_1' \cdot 0 + C_2' \cdot e^x + C_3'\, e^{-x} = 2x + 1 - 4\cos x + 2e^x \end{array} \right. \qquad (32)$$

Determinant $$\begin{vmatrix} 1 & e^x & e^{-x} \\ 0 & e^x & -e^{-x} \\ 0 & e^x & e^{-x} \end{vmatrix} = 2.$$

$$C_1' = \frac{\begin{vmatrix} 0 & e^x & e^{-x} \\ 0 & e^x & -e^{-x} \\ 2x+1-4\cos x+2e^x & e^x & e^{-x} \end{vmatrix}}{2} = \frac{1}{2}(2x + 1 - 4\cos x + 2e^x)(-2) = -2x - 1 + 4\cos x - 2e^x, \qquad (33a)$$

$$C_2' = \frac{1}{2}\begin{vmatrix} 1 & 0 & e^{-x} \\ 0 & 0 & -e^{-x} \\ 0 & 2x+1-4\cos x+2e^x & e^{-x} \end{vmatrix} = \frac{1}{2}\, e^{-x}(2x+1-4\cos x+2e^x), \qquad (33b)$$

$$C_3' = \frac{1}{2}\begin{vmatrix} 1 & e^x & 0 \\ 0 & e^x & 0 \\ 0 & e^x & 2x+1-4\cos x+2e^x \end{vmatrix} = \frac{1}{2}\, e^x(2x+1-4\cos x+2e^x). \qquad (33c)$$

We integrate out each equation to obtain

$$\begin{cases} C_1(x) = -x^2-x+4\sin x-2e^x+k_1 \quad ; \quad k_1 = \text{const.} \\ C_2(x) = -xe^x - \frac{3}{2}e^{-x} + e^{-x}\cos x - e^{-x}\sin x + x + k_2 \; ; \quad k_2 = \text{const.} \\ C_3(x) = xe^x - \frac{1}{2}e^x - e^x\cos x - e^x\sin x + \frac{1}{2}e^{2x} + k_3 ; \quad k_3 = \text{const.} \end{cases} \qquad (34)$$

It follows that

$$\begin{aligned} u_p(x) = {} & (-x^2-x+4\sin x-2e^x+k_1)\cdot 1 + (-xe^x - \tfrac{3}{2}e^{-x} + e^{-x}\cos x - e^{-x}\sin x + x + k_2)\, e^x \\ & + (xe^x - \tfrac{1}{2}e^x - e^x\cos x - e^x\sin x + \tfrac{1}{2}e^{2x} + k_3)\, e^{-x} \\ = {} & -x^2-x+2\sin x+xe^x + (k_1-1)\cdot 1 + (k_2-\tfrac{3}{2})\cdot e^x + k_3\cdot e^{-x}, \end{aligned} \qquad (35)$$

which recovers the particular solution of Ex. 3 by setting

$k_1 = 1$, $k_2 = 3/2$, $k_3 = 0$.

Note that this method, in fact, recovers the general solution to Eq. (10) (k_1, k_2, k_3 in the above example are arbitrary).

$\boxed{\text{Case 5:}}$ Linear ODE with <u>non-constant</u> coefficients, i.e.

$$u^{(n)}(x) + a_{n-1}(x)\,u^{(n-1)}(x) + a_{n-2}(x).u^{(n-2)}(x) + \dots + a_1(x)\,u'(x) + a_0(x)u(x) = g(x). \qquad (36)$$

In principle, this equation may not be completely solvable. Eq. (11) holds in this case as well.

- If we know <u>n</u> <u>linearly independent</u> solutions $u_k(x)$ of the homogeneous equation (with $g(x)\equiv 0$), we can find a particular solution to (36) by the method of variation of parameters (see pp. 6-8 above).

- If we know <u>one</u> solution of the homogeneous equation, we can <u>reduce</u> (36) from order n <u>to order $(n-1)$</u>. This method is called <u>reduction of order</u>.

Suppose that we know $u_h(x) = u_1(x)$ as <u>one</u> "homogeneous" solution to (36) (i.e., with $g\equiv 0$). We reduce the order by setting

$$u(x) = v(x)\,u_1(x) \qquad (37)$$

and obtaining an $(n-1)$-order ODE <u>for $v'(x)$</u>.

<u>Example 5:</u> Solve $x^2u'' + x(x-1)u' - (x-1)u = x^2e^{-x}$, $x>0$, (38)

by knowing that a solution to $x^2u''+x(x-1)u'-(x-1)u=0$ is $u_1 = x$.

Let $\underline{u(x) = x\,v(x)}$ (reduction for of order). (39)

$$\therefore \quad u'(x) = v + xv', \qquad u'' = 2v' + xv''. \qquad (40)$$

$$\overset{(38)}{\Rightarrow} \quad 2x^2v' + x^3v'' + x(x-1)(\cancel{v}+xv') - \cancel{(x-1)xv} = x^2e^{-x} \qquad (v\text{'s cancel})$$

$$\Rightarrow \quad x^3v'' + x^2(x+1)v' = x^2e^{-x} \quad \overset{x\neq 0}{\Longrightarrow} \quad xv'' + (x+1)v' = e^{-x}. \qquad (41)$$

Let $\underline{v' \equiv w}$: $\quad xw' + (x+1)w = e^{-x} \Leftrightarrow w' + \overbrace{\left(1+\frac{1}{x}\right)}w = \dfrac{e^{-x}}{x}$. (42)

According to (5), the solution to (42) is

$$w(x) = \exp\left[-\int dx\,\left(1+\frac{1}{x}\right)\right] \cdot \left\{ \int dx\, \frac{e^{-x}}{x}\, e^{\int dx\,(1+\frac{1}{x})} + K_1 \right\}$$

$$= \frac{e^{-x}}{x}\,(x+K_1)\,, \qquad K_1 = \text{const.}$$

$$\Rightarrow \quad v' = w \Rightarrow \quad v = \int dx\, w(x) + K_2 = -e^{-x} + K_1 \int dx\, \frac{e^{-x}}{x} + K_2\,.$$

$$\therefore \quad u(x) = x\,v(x) = -x\,e^{-x} + K_1 x \int dx\, \frac{e^{-x}}{x} + K_2 x\,, \qquad K_1; K_2 = \text{const.} \qquad (43)$$

[Note: The indefinite integral $\int dx\, \frac{e^{-x}}{x}$ cannot be calculated in terms of simple, elementary fcns.]

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$\boxed{\text{Particular case of (36):}}$

$$x^n u^{(n)}(x) + b_{n-1} x^{n-1} u^{(n-1)}(x) + \ldots + b_1 x\, u'(x) + b_0 u(x) = g(x). \qquad (44)$$

This eqn. is called <u>equidimensional</u>. It can be solved exactly by the <u>change of variable</u>

(Coefficients $b_j x^j$ have <u>powers</u> of x equal to order of multiplying derivatives)

$$x = e^z, \qquad z: \text{new variable.} \qquad (45)$$

Then (44) is transformed into a linear ODE with <u>constant</u> coefficients in z.

Just notice that

$$x^m u^{(m)}(x) = \frac{d}{dz}\left(\frac{d}{dz}-1\right)\left(\frac{d}{dz}-2\right)\cdots\left(\frac{d}{dz}-m+1\right)u. \qquad (46)$$

<u>Example 6</u>: Solve $x^2u'' - 2xu' + 2u = x^2+2.$ (47)

This is an equidimensional equation. With $x=e^z$, the transformed equation is

$$\mathcal{U}''(z) - 3\,\mathcal{U}'(z) + 2\,\mathcal{U}(z) = e^{2z} + 2\,, \qquad (48)$$

where $\mathcal{U}(z) \equiv u(x(z))$. The solution to (48) can be found by methods described in <u>Case 4</u>:

$$\mathcal{U}(z) = c_1 e^z + c_2 e^{2z} + z e^{2z} + 1 \quad \overset{z=\ln x}{\Longrightarrow} \quad u(x) = c_1 x + c_2 x^2 + x^2 \ln x + 1. \qquad (49)$$

$$c_1, c_2 = \text{const.}$$

$\boxed{\text{Case } 6:}$ Some solvable nonlinear ODE (list is not exhaustive)

(A) $$P(x,u)\,dx + Q(x,u)\,du = 0 \quad ; \quad u = u(x). \qquad (50)$$

In principle, this equation is not solvable. A sufficient condition that enables one to find explicit or implicit solutions is
$$\frac{\partial P}{\partial u} = \frac{\partial Q}{\partial x}. \qquad (51)$$

In this case, (50) is called exact. Then there exists a function w such that
$$dw = P(x,u)\,dx + Q(x,u)\,du = 0 \Rightarrow w = K = \text{const}. \qquad (52)$$

So, it suffices to find $w = w(x,u)$ as a function of x and u and find the relation between x and u **via** (52).

Example 7 : Solve $$u'(x) = \frac{1+u^2+3x^2u}{1-2xu-x^3} \quad ; \quad u' = \frac{du}{dx}. \qquad (53)$$

This equation is written as
$$\underbrace{(3x^2u+u^2+1)}_{P(x,u)}\,dx + (\underbrace{x^3+2xu-1}_{Q(x,u)})\,du = 0. \qquad (54)$$

Notice that $$\frac{\partial P}{\partial u} = 3x^2+2u = \frac{\partial Q}{\partial x} \quad ; \quad \text{so (51) is satisfied.}$$

We require that there exists $w = w(x,u)$ such that
$$dw = P(x,u)\,dx + Q(x,u)\,du = \overbrace{\frac{\partial w}{\partial x}\,dx + \frac{\partial w}{\partial u}\,du}^{\text{Taylor expansion in } (x,u)}. \qquad (55)$$

It follows that $$\frac{\partial w}{\partial x} = P(x,u) = 3x^2u+u^2+1 \xRightarrow{\text{integrate in } x} w(x,u) = x^3u + xu^2 + x + \overset{\text{arbitrary}}{f(u)}. \qquad (56)$$

From (56), $\frac{\partial w}{\partial u} = x^3+2xu+f'(u)$ which must be set equal to $Q(x,u)$ from (55):
$$x^3+2xu+f'(u) = x^3+2xu-1 \Rightarrow f'(u) = -1 \Rightarrow f(u) = -u+K$$

Finally, $$w(x,u) = x^3u+xu^2+x-u+K$$

So, the solution to (53) is written as

$$w = \text{const.} \quad \Longleftrightarrow \quad x^3 u + x u^2 + x - u = c = \text{const.}$$

Ⓑ Bernoulli's equation: $\quad u'(x) + P(x)\, u(x) = Q(x)\, u(x)^n \quad (n \neq 1). \qquad (57)$

This equation is written in the form

$$u^{-n}\, u' + P(x)\, u^{-n+1} = Q(x) \quad \Longrightarrow \quad \frac{-1}{n-1}\left(u^{-n+1}\right)' + P(x)\cdot\left(u^{-n+1}\right) = Q(x), \qquad (58)$$

which becomes linear (in v) via $\quad v \equiv u^{-n+1}$.

Ⓒ Riccati's equation: $\quad u'(x) + P(x)\, u(x) + Q(x)\, u(x)^2 = R(x). \qquad (59)$

This equation is reduced to a linear 2nd-order eqn. (in new fcn w) via

$$u = \frac{1}{Q} \underbrace{\frac{d}{dx} \ln w}_{\text{“logarithmic derivative”}} = \frac{w'}{Qw}. \qquad (60)$$

Indeed,

$$u' = \frac{w''}{Qw} - \frac{w'^2}{Qw^2} - \frac{Q'w'}{Q^2 w}.$$

$$(59): \quad \frac{w''}{Qw} - \cancel{\frac{w'^2}{Qw^2}} - \frac{Q'w'}{Q^2w} + \frac{Pw'}{Qw} + \cancel{Q\left(\frac{w'}{Qw}\right)^2} = R \qquad \text{(nonlinear terms cancel!)}$$

ODE for w: $\Longrightarrow$

$$w'' + \left(P - \frac{Q'}{Q}\right) w' - RQw = 0. \qquad (61)$$

[Note: If we know one solution $u = u_1(x)$ of (59), the substitution $u = u_1 + \frac{1}{v}$ leads to the linear, 1st-order ODE for v $\quad v' - (P + 2Qu_1)\, v = Q$.]

Example 8 : Solve the ODE $\quad x^2 u' + xu + x^2 u^2 = 1$. given that $u_1 = \frac{1}{x}$ is a solution

It is left as an exercise. Notice that having $u_1(x) = \frac{1}{x}$ reduces the ODE to a linear, 1st-order one.

Answer: $\quad u(x) = \frac{1}{x}\,\frac{x^2 - c}{x^2 + c}, \quad c = \text{const.}$

$\boxed{\text{Case } 6':}$ In general, when you encounter a nonlinear ODE, try to use your ingenuity, inventing tricks of your own if possible.

Example 9 : Solve $u''+u^3=u$ (autonomous; lacks x) (62)

Multiply both sides by u': $u''u'+u^3u'=uu'$

$$\Rightarrow \frac{d}{dx}\left(\frac{1}{2}u'^2+\frac{1}{4}u^4\right)=\frac{d}{dx}\left(\frac{1}{2}u^2\right) \Rightarrow u'^2+\frac{1}{2}u^4-u^2=C=\text{const.}$$

$$\Rightarrow u'=\pm\sqrt{C+u^2-\frac{1}{2}u^4}, \qquad (63)$$

which is in separable form and can be integrated out directly.

Example 10 : Solve $u''=x\,(u')^3$ (lacks variable u). (63)

Set $p(x)=p\equiv u' \Rightarrow p'=u''$. (64)

$$\overset{(63)}{\Rightarrow} \quad p'=xp^3 \Rightarrow p(x)=\frac{\pm 1}{\sqrt{k_1^2-x^2}}, \quad k_1=\text{const.}$$

$$\therefore \quad \frac{du}{dx}=\frac{\pm 1}{\sqrt{k_1^2-x^2}} \overset{\text{separable}}{\Rightarrow} u(x)=\pm\arcsin\left(\frac{x}{k_1}\right)+k_2 .$$

Example 11 : Solve $u^2u''=u'^2$ (lacks variable x) (64)

This ODE lacks the variable x. Set

$p=u'=p(u)$, i.e., view u' as a fnc of u!

Then $u''=\frac{dp}{dx}=\frac{dp}{du}u'=p'p$.

$$\overset{(64)}{\Rightarrow} \quad u^2p'p=p^2 \Rightarrow p\,(u^2p'-p)=0 \Rightarrow \begin{cases} p=0 \Rightarrow u=c_1 \\ \text{or} \\ u^2p'=p \Rightarrow p=c_2\,e^{-1/u}=\frac{du}{dx} \end{cases}$$

The second alternative ($p=c_2e^{-1/u}$) contains the first one ($p=0$) as a special case ($c_2\equiv 0$)

General solution : $\int du\, e^{1/u}=c_2x+c_3$ (integral may not be calculable in terms of elementary functions).

Example 12 : Solve $u'^2 u'' = 1 + u'^2$. (65)

This ODE lacks both x and u. As in Ex. 10, set

$$p(x) = p \equiv u' \overset{(65)}{\rightarrow} p^2 \frac{dp}{dx} = 1+p^2 \Longrightarrow \frac{p^2}{1+p^2}\, dp = dx \Rightarrow x = p - \arctan(p) + K_1. \quad (66)$$

We thus eliminate x by the relation

$$p = \frac{du}{dx} \Rightarrow du = p\, dx = p \frac{dx}{dp} \cdot dp = \frac{p^3}{1+p^2}\, dp$$

$$\Longrightarrow \quad u = \frac{1}{2} p^2 - \frac{1}{2} \ln(p^2+1) + K_2. \quad (67)$$

For each x , p is determined via (66) and then used to determine u via (67); hence, (67) is viewed as a parametric equation for u.