M is found by substitution in PDE: $(4-n^2)M = -A^2 b_n \Rightarrow \boxed{M = \frac{A^2 b_n}{n^2-4}}$

$\Rightarrow \boxed{F = -\frac{A^2 b_n}{n^2-4}}$

$\Rightarrow \quad a_n(x) = \frac{A^2 b_n}{n^2-4}\left(e^{-2x} - e^{-nx}\right)$ $\quad b_n = \begin{cases} 0 & n: \text{even} \\ \frac{4}{\pi}\frac{n^2-2}{(n^2-4)n} & n: \text{odd} \end{cases}$

$$u^{(1)}(x,y) = \frac{4A^2}{\pi}\sum_{n:\text{odd}} \frac{n^2-2}{(n^2-4)^2}\frac{1}{n}\left(e^{-2x} - e^{-nx}\right)\sin(ny)$$

$$u(x,y) = u^{(0)}(x,y) + \varepsilon u^{(1)}(x,y) + \dots \qquad |\varepsilon| \ll 1$$

Issues about convergence:

- Does the series $u = u^{(0)} + \varepsilon u^{(1)} + \dots + \varepsilon^n u^{(n)} + \dots$ converge?

Answer: Yes, by theory of integral equations.
↳ for sufficiently small ε, $|\varepsilon| < |\varepsilon_0|$

- More generally, do series of such form converge?

Answer: No, in principle; They may be divergent, yet assymptotic.

(assymptotic series: you fix n and you look if your finite sum is close to the true u if $\varepsilon \ll 1$)

May 5, 2004 Lecture 26

- Quiz 2: Mon. May 10, pick up in class
  to be returned Wed. May 12 before class
- Review Session: PR1 4-5:30 pm

Review regular perturbations for PDE
(advice: have eqns. in non-dimensional form)

General problem: PDE: $\begin{cases} F(x, x_2, \dots x_n, \text{deriv. of } u; \varepsilon) = 0, & |\varepsilon| \ll 1 \\ + \text{ condns} \end{cases}$

⚠ existence of solution for $\varepsilon=0$ is necessary but not sufficient condition for Reg. Pert. to work

Steps: (i) Set $\varepsilon=0$: verify that a solution ~~exist~~ that satisfies $\underbrace{\text{PDE + given conditions}}_{\text{all!}}$ exists.

↱ not derivative!

(ii) Apply series $u = u^{(0)} + \varepsilon u^{(1)} + \varepsilon^2 u^{(2)} + \dots + \varepsilon^n u^{(n)} + \dots$

where $u^{(k)} = O(1)$, $|\varepsilon| \ll 1$ ↳ we assume power series in ε $\Rightarrow$ analytic solution in ε

- substitute in PDE and condns; derive equations for $u^{(k)}$, $k=1,2,\dots$ given $u^{(0)}$
- solve equations for $u^{(k)}$ successively.

Usually series converges for suff. small ε

Ex. Schrödinger equation:
$$\begin{cases} -\nabla^2 u + \underset{\text{given}}{V(\vec r)}\, u = \underset{\text{const}}{E}\, u \\ u \to 0, \ |\vec r| \to \infty \end{cases}$$

where $V(\vec r) = V_0(\vec r) + \varepsilon W(\vec r)$, $|\varepsilon| \ll 1$

Claim: if $\left|\dfrac{\varepsilon W(\vec r)}{V_0(\vec r)}\right| \to 0$ as $\varepsilon \to 0$ uniformly in $\vec r$ then $u = u^{(0)} + \varepsilon u^{(1)} + \dots$ convergent

Singular Perturbation Theory:

Symptom: Solution to a problem doesnot exist for $\varepsilon=0$.

(A) ODEs

Example:
$$\begin{cases} \varepsilon u'' + u' = 1 \qquad u = u(x) \qquad 0<x<1 \\ u(0) = 0 = u(1) \end{cases}$$

Exact solution:
$$u(x) = \underbrace{x-1}_{\alpha(x)} + \underbrace{\frac{e^{-x/\varepsilon} - e^{-1/\varepsilon}}{1-e^{-1/\varepsilon}}}_{\beta(x/\varepsilon)}$$

depends on x — depends on x/ε scaled by $1/\varepsilon$

we have $\alpha(1)=0$

$p(\frac{x}{\varepsilon})$ for $\varepsilon \ll 1 \approx e^{-x/\varepsilon}$

solution is $u(x) \approx x - 1 + e^{-x/\varepsilon}$

- $\frac{x}{\varepsilon} \gg 1$: $e^{-x/\varepsilon} \approx 0$, $u(x) \approx x-1$: solution for $\varepsilon = 0$ of the ODE $\varepsilon u'' + u' = 1$

"OUTER SOLUTION"

the term $\varepsilon u''$ was neglected

$\underline{\varepsilon = 0}$: $u' = 1$

$u = x + C$, $u(1) = 0$

$\Rightarrow \boxed{u = x - 1}$

two ways to obtain it: $\varepsilon u''$ neglected; take $\frac{x}{\varepsilon} \gg 1$

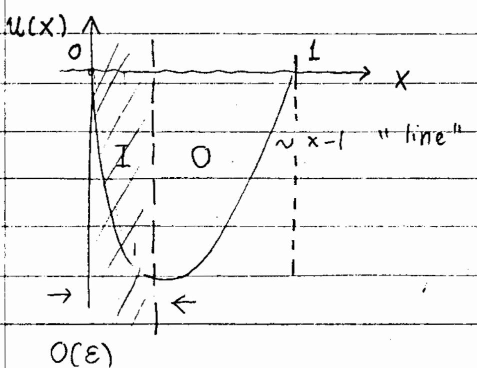
- $\frac{x}{\varepsilon} = O(1) \Rightarrow x = O(\varepsilon)$: $u \approx -1 + e^{-x/\varepsilon}$

x neglected; can not be neglected $\frac{x}{\varepsilon} = O(1)$

"INNER SOLUTION"

- x scales with some power of ε $\quad \xi = \frac{x}{\varepsilon} = O(1)$
- $\varepsilon u''$ can NOT be neglected

Look Hinch for BL theory.



The region near $x = 0$ (in this Ex.) where $u(x)$ varies rapidly (and x scales with a function of ε) is called BOUNDARY LAYER

Inner solution:	$-1 + e^{-x/\varepsilon}$	(I)	$\frac{x}{\varepsilon} \to +\infty$ $((\frac{x}{\varepsilon}) \gg 1)$ $\to$	≈ -1	Agree!
Outer solution:	$x - 1$	(O)	$x \to 0$ $\to$	≈ -1	

$\hookrightarrow$ overlap region

Any deviation from the overlap value in the inner solution should decay exponentially.

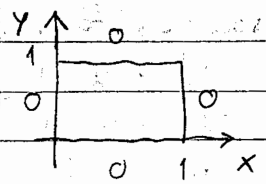
$\hookrightarrow$ important

(B) Case of PDE's

singular perturbation: ε multiplies the highest derivative

Example: $\varepsilon \nabla^2 u + u_x = y(1-y^2)$, $u = u(x,y)$

$|\varepsilon| << 1$



(i) Try to set $\varepsilon = 0$:

$u_x = y(1-y^2) \Rightarrow u(x,y) = xy(1-y^2) + C(y)$

we choose that

- suppose $u(x=0,y) = 0 \Rightarrow C = 0$
- suppose $u(x=1,y) = 0 \Rightarrow C = -y(1-y^2)$

$\Rightarrow$ we can not satisfy both conditions. at $x = 0, 1$.

$\rightarrow$ indication for singular perturbation theory

(ii) the solution for $\varepsilon = 0$ can be written as

$u^{(0)} = y(1-y^2)\,[\,x - A(y)\,]$ satisfies conditions at $y = 0, 1$

the problem is with the condition in x

We seek (possible) boundary layers at $x = 0$ and $x = 1$.

- $\boxed{\text{Try } x = 0}$: How does the thickness scale?

$\rightarrow$ assume that x scales by ε (as in the ODE example)

$\xi = \frac{x}{\varepsilon}$: $u(x,y) = h(x,y) + p(x,y;\varepsilon)$

solution for $\varepsilon = 0$: $u^{(0)}$

remainder, takes care of condition at $x = 0, 1$.

$h(x,y) = y(1-y^2)\,[\,x - A(y)\,]$

Find PDE for $p = p(\xi, y)$

$u_{xx} = h_{xx} + p_{xx} = \frac{1}{\varepsilon^2} p_{\xi\xi}$, $u_x = y(1-y^2) + \frac{1}{\varepsilon} p_\xi$, $u_{yy} = h_{yy} + p_{yy}$

$$\varepsilon h_{yy} + \varepsilon p_{yy} + \frac{1}{\varepsilon} p_{\xi\xi} + \underbrace{h_x + \frac{1}{\varepsilon} p_\xi}_{\text{cancell}} = y(1-y^2)$$

$$\Rightarrow \quad \frac{1}{\varepsilon} p_{\xi\xi} + \frac{1}{\varepsilon} p_\xi + \varepsilon p_{yy} = -\varepsilon h_{yy}$$

for a BL at $x=0$: $\frac{x}{\varepsilon} = \xi = O(1)$ & $\varepsilon \ll 1$

$$\Rightarrow \quad p_{\xi\xi} + p_\xi + \cancel{\varepsilon^2 p_{yy}} = \cancel{-\varepsilon^2 h_{yy}}$$

assume $p_\xi, p_{\xi\xi}, p_y, p_{yy} = O(1)$ so neglect the ε^2 terms.

PDE ⇒ ODE

$$p_{\xi\xi} + p_\xi = 0 \Rightarrow \text{ODE} \Rightarrow p(\xi, y) = B(y) e^{-\xi} + D(y)$$

$$u(x,y) \simeq y(1-y^2)\left[x - A(y)\right] + B(y) e^{-x/\varepsilon} + D(y)$$

$$= y(1-y^2)\left[x - A(y)\right] + B(y) \boxed{e^{-x/\varepsilon}}$$

↳ decays exponentially

Apply conditions:

$$\begin{cases} u(x=0,y) = 0 \\ u(x=1,y) = 0 \end{cases}$$

↳ we try to expand the sol. for BL to the whole region

$$\Rightarrow \begin{cases} -y(1-y^2) A(y) + B(y) = 0 \\ y(1-y^2)\left[1 - A(y)\right] + B(y) e^{-1/\varepsilon} = 0 \end{cases} \qquad A = \frac{1}{1-e^{1/\varepsilon}} \approx 1, \quad B = \frac{y(1-y^2)}{1-e^{1/\varepsilon}} \approx (y(1-y^2))$$

$$\Rightarrow \quad u(x,y) \approx y(1-y^2)\left[x - 1 + e^{-x/\varepsilon}\right]$$

↳ if we try a BL at $x=0$ we have an exp. decaying solution that we expand to $x=1$

Check:

condns at $y=0,1 \Rightarrow u=0$

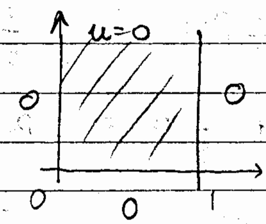
at $x=1 \Rightarrow u(1,y) = y(1-y^2) e^{-1/\varepsilon} \approx 0$ (exp. small)

at $x=0 \Rightarrow u(0,y) = 0$

PDE : $\left[x - 1 + e^{-x/\varepsilon}\right](-6y) + \frac{1}{\varepsilon^2} e^{-x/\varepsilon} y(1-y^2) + y(1-y^2)\left[\cancel{1} - \frac{1}{\varepsilon} e^{-x/\varepsilon}\right] = \cancel{y(1-y^2)}$

⇒ PDE is satisfied at order ε

Example:



$\varepsilon\nabla^2 u - u_x + 2u = 1$

$|\varepsilon| \ll 1$

(i) $\varepsilon = 0$: $\quad -u_x + 2u = 1$

$\Rightarrow \quad u^{(0)}(x,y) = \frac{1}{2}\left[A(y)e^{2x} + 1\right]$

can not satisfy both conditions at $x = 0, 1$

(ii) singular perturbation theory : look for BL

$u(x,y) = \underbrace{h(x,y)}_{\varepsilon = 0} + p(x,y;\varepsilon)$

PDE for p: $\quad \boxed{\varepsilon\nabla^2 p - p_x + 2p = -\varepsilon\nabla^2 h}$

Look for boundary layers:

① $\underline{x=0}$: $\quad \xi = \frac{x}{\varepsilon^\nu}, \quad \nu > 0 \quad$ we want to find ν

$p = p(\xi, y)$; $\xi = O(1)$ inside BL

PDE for p:

$$\underbrace{p_{\xi\xi}}_{O(1)} + \varepsilon^{2\nu}\underbrace{p_{yy}} - \varepsilon^{\nu-1}\underbrace{p_\xi}_{O(1)} + 2\varepsilon^{2\nu-1}\underbrace{p}_{O(1)} = -\varepsilon^{2\nu}\underbrace{\cancel{\nabla^2 h}}_{O(1)}$$

can not have BL at $x = 0$

② $\underline{x=1}$ $\quad \xi = \frac{1-x}{\varepsilon^\nu} \quad \nu > 0$

$$p_{\xi\xi} + \varepsilon^{2\nu} p_{yy} + \varepsilon^{\nu-1} p_\xi + 2\varepsilon^{2\nu-1} p = -\varepsilon^{2\nu}\cancel{\nabla^2 h}$$