18.306: Advanced Partial Differential Equations with Applications
M.I.T. Department of Mathematics **Spring 2004**
Practice Set 2 **Handed out**: Wednesday, 02/25/04
Due: NEVER

You are NOT required to return solutions. Some of these problems may be given in Quiz 1. The description of the references given in parentheses can be found in the Bibliography for 18.306.

11. (Debnath, Prob. 3.6.17, p. 132:) Solve the following equations:

    (a) $(y+u)u_x + (x+u)u_y = x+y$. **Hint:** Use $\frac{d(x+y+u)}{2(x+y+u)} = \frac{d(y-u)}{-(y-u)} = \frac{d(u-x)}{-(u-x)}$ etc. (why?)

    (b) $xu(u^2+xy)u_x - yu(u^2+xy)u_y = x^4$.

12. (Debnath, Prob. 3.6.20, p. 132:) Find the general solution of the equation $y\,u_x - 2xy\,u_y = 2xu$. In particular, determine the solution that satisfies $u(0,y) = y^3$. [**Ans.** General solution: $yu = F(x^2+y)$.]

13. [Levine, Chap. 11, Prob. 7, pp. 126, 127.] (a) Confirm that $u(x,y;c_1,c_2) = c_1^2 x + c_1 y + c_2$ represents a "complete integral" of the PDE $p = q^2$ ($p = u_x$, $q = u_y$), where c_1 and c_2 are arbitrary constants.

    (b) For (a), consider the data $x(s) = 0, y(s) = s$, and $u(s) = s^2$, where s is a real parameter. Determine c_1 and c_2 in terms of s, and then give the solution u as a function of x and y only.

14. (Levine, Chap. 11, p. 126.) Describe the solution to the PDE $p - q = a\,pq$ ($p = u_x$, $q = u_y$), where a is a real constant.

15. In materials science, the evolution of surfaces of materials is often studied by use of PDE. In the case of evaporation dynamics, where material evaporates to the environment, the surface height $h = h(x,y,t)$ relative to the xy-plane of reference satisfies a complicated 2nd-order PDE. For simplicity, it is assumed that axisymmetry holds, i.e., h depends only on the polar distance $r = \sqrt{x^2+y^2} \geq 0$ and time $t > 0, h = h(r,t)$; it is also assumed that h satisfies the simplified PDE $h_t = (A/r)h_r$, where A is a positive constant. Show that any initial smooth, axisymmetric bump $h(r,0) = H(r)$, where $H(r)$ is monotonically decreasing to 0, finally shrinks to zero as $t \to +\infty$.[**Ans:** By use of characteristics, $h(r,t) = H(\sqrt{2At + r^2})$.]

16. (Levine, Chap. 9, Prob. 11, p. 106.) The necessary and sufficient condition that $\mu(x,y)$ is an integrating factor of the ODE

$$a(x,y)\,dx + b(x,y)\,dy = 0 \qquad (1)$$

is the following PDE satisfied by μ (why?):

$$\frac{\partial(\mu a)}{\partial y} = \frac{\partial(\mu b)}{\partial x}. \qquad (2)$$

Take $a(x,y) = 3xy + 2y^2$, $b(x,y) = 3xy + 2x^2$ and show that an integral of (2) is

$$\phi(x, y, \mu) = \mu \sqrt{x+y} = \text{const.},$$

and thus obtain the general solution to (1). **Note:** The knowledge of only <u>one</u> complete integral of (2) suffices to solve (1).

17. (Levine, Chap. 9, Prob. 12, pp. 106, 107.) Consider the quasi-linear PDE

$$uu_x + u_y = 0, \qquad -\infty < x < \infty,\ y > 0,$$

with the initial data $u(x,0) = \text{sech} x$, $-\infty < x < \infty$. Show that a differentiable ("regular") solution exists so long as $y < y^*$, where y^* is the minimum value of the function

$$\frac{\cosh^2 s}{\sinh s}, \qquad s > 0.$$

Describe (with a sketch) the solution $u(x,y)$ for $y > y^*$. **Recall:** $\text{sech}\, x = 1/\cosh x$.

18. Use characteristics in order to solve the system of PDE

$$u_t + v_x = 0, \quad v_t + u_x = 0$$

with the initial data $u(x,0) = f(x)$ and $v(x,0) = g(x)$.

19. (Debnath, Prob. 6.11.12, p. 257.) For one-dimensional anisentropic fluid flow, the Euler equations of motion read as

$$p_t + up_x + c^2 \rho\, u_x = 0,$$
$$u_t + uu_x + + \frac{1}{\rho} p_x = 0,$$
$$S_t + uS_x = 0,$$

where $p = f(\rho, S)$ is the pressure, u is the velocity, ρ is the fluid density, and S is the entropy. Show that this system of PDE has three families of characteristics given by

$$C_0 : \ \frac{dx}{dt} = u, \qquad C_+ : \ \frac{dx}{dt} = u + c, \qquad C_- : \ \frac{dx}{dt} = u - c,$$

where $c^2 \equiv (\partial p/\partial \rho)_x$ =const.

20. A *uniform* solid sphere with radius a is kept initially ($t = 0$) at constant temperature $T = T_0$. At all later times $t > 0$ the surface temperature of the sphere, at $r = a$, is maintained at $T = 0$, where r is the spherical coordinate with origin at the center of the sphere. Given that the temperature distribution inside the sphere satisfies the diffusion equation, $T_t = \nu \nabla^2 T$, where $\nu > 0$, define and calculate the average temperature T_{ave} inside the sphere for $t > 0$. Derive an approximate formula for T_{ave} for sufficiently long times t.