March 29, 2004 Lecture 14

Homework 3 : Due Monday April 5

"Natural frequencies" (of homogeneous PDE)

<u>Example</u> : Vibrating String

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 , & 0 < x < L \\ u(0,t) = 0 = u(L,t) & : \text{homogeneous bc's} \\ + \text{ IC's} \end{cases}$$

Separation of variables :

$u(x,t) = X(x)\, T(t) \Rightarrow T(t) = A \cos \omega t + B \sin \omega t$

$X(x) = C \cos \frac{\omega}{c} x + D \sin\left(\frac{\omega}{c} x\right)$

Homogeneous bc's:

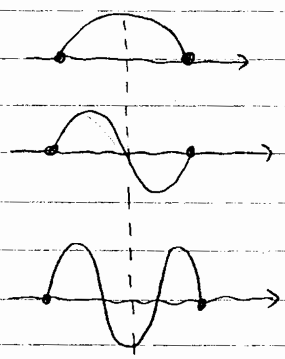
$$X(0) = X(L) = 0 \Rightarrow \begin{cases} C = 0 \\ \omega = \omega_n = \frac{n\pi c}{L} , \quad n = 1, 2, \ldots \end{cases} \text{"natural frequencies"}$$

n = 1

n = 2

⋮

n = 3



" modes of vibration"

(higher n: more oscillations)

For IC's $u(x,t) = \sum_{\{\omega_n\}} X_n(x)\, T_n(t)$

Sturm-Liouville problem :

$$\begin{cases} \frac{d}{dx}\left[p(x) \frac{d\phi}{dx}\right] + \left[q(x) + \mu S(x)\right] \phi(x) = 0 , & a < x < b \\ + \text{ homogeneous bc's} \end{cases}$$

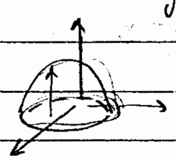
Conclusions: μ in $\underbrace{\{\mu_n\}}^{\text{eigenvalues}}$, ϕ in $\{\phi_n\}$

$\{\phi_n\}$ complete set

orthogonal $\int_a^b dx\, \underbrace{S(x)}_{\text{weight}}\, \phi_n(x)\, \phi_m(x) = 0, \quad \mu_n \neq \mu_m$

Ex. Vibrating circular membrane

$u = u(\rho, \theta, t) \quad 0<\rho<a$



$$\begin{cases} u_{tt} - c^2 \nabla^2 u = 0 \\ u(\rho=a, \theta, t) = 0, \quad u: \text{bounded} \quad : \text{homogeneous bc's} \\ u_t(\rho, \theta, t)\big|_{t=0} = 0 \\ u(\rho, \theta, t)\big|_{t=0} = f(\rho, \theta) \end{cases}$$

Apply separation of variables: $u(\rho, \theta, t) = \Lambda(\rho, \theta)\, T(t)$

$\Lambda T'' - c^2 T \nabla^2 \Lambda = 0 \quad \Rightarrow \quad c^2 \dfrac{\nabla^2 \Lambda}{\Lambda} = \dfrac{T''}{T} = -\lambda$ (last time we show λ is positive)

- $T(t) = A \cos\sqrt{\lambda}\, t + B \sin\sqrt{\lambda}\, t$, $\sqrt{\lambda} = \omega$
- $c^2 \dfrac{\nabla^2 \Lambda}{\Lambda} = -\omega^2 \quad \Rightarrow \quad \nabla^2 \Lambda + \left(\dfrac{\omega}{c}\right)^2 \Lambda = 0$ Helmholtz equation

$k = \omega/c$ wave number

Separation of variables: $\Lambda(\rho, \theta) = R(\rho)\, \Theta(\theta)$

$$\nabla^2 \Lambda = \left(\frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial \theta^2}\right)(R\Theta) = \Theta\left(R'' + \frac{R'}{\rho}\right) + \frac{R}{\rho^2}\Theta''$$

PDE for $\Lambda \Rightarrow \quad \dfrac{1}{R}\left(\rho^2 R'' + \rho R'\right) + \dfrac{\Theta''}{\Theta} + (k\rho)^2 = 0$

$$\frac{1}{R}\left(\rho^2 R'' + \rho R'\right) + (k\rho)^2 = -\frac{\Theta''}{\Theta} = \alpha = \text{const}$$

$$① \begin{cases} \Theta'' + \alpha\,\Theta(\theta) = 0 \\ \theta \in (0, 2\pi) \\ \Theta(0) = \Theta(2\pi) \\ \Theta'(0) = \Theta'(2\pi) \end{cases} \text{periodicity} \qquad ② \begin{cases} \rho^2 R'' + \rho R' + \left[(k\rho)^2 - \alpha\right] R = 0 \\ 0<\rho<a \\ R(a) = 0 \\ R(0) \text{ bounded} \end{cases}$$

It's a Sturm-Liouville problem.

→ Solve problem 1: $\Theta(\theta) = A_1 \cos\sqrt{\alpha}\,\theta + A_2 \sin\sqrt{\alpha}\,\theta$

or $\Theta(\theta) = e^{\pm i\sqrt{\alpha}\,\theta}$

we want Θ periodic $\Rightarrow \alpha = n^2$ since we want $\sqrt{\alpha} = n$

$n = 0, 1, 2, \ldots$

→ Solve problem 2:

$\rho^2 R'' + \rho R' + \left[(k\rho)^2 - n^2\right] R = 0$

it should be a S-L problem $\quad \left(\rho R'' + R'\right) + \left(k^2 \rho - \dfrac{n^2}{\rho}\right) R = 0$

$$\frac{d}{d\rho}\left(\rho R'\right) + \left(k^2\rho - \frac{n^2}{\rho}\right) R = 0$$

it is a S.-L. problem : $\phi = R$, $x = \rho$

with $p = \rho$, $q = -\frac{n^2}{\rho}$, $k^2 = \mu$, $s = \rho$

(n is given from periodicity, it is not our μ !)

$\rho^2 R'' + \rho R' + (k^2\rho^2 - n^2) R = 0$

Recall :

Bessel ODE $x^2 y''(x) + x y'(x) + (x^2 - \beta^2) y(x) = 0$, $\beta > 0$

solutions are Bessel functions:

$y(x) = C_1 J_\beta(x) + C_2 Y_\beta(x)$

Bessel fns ↑ OK at 0 — Neumann fcns ↑ blows up at zero

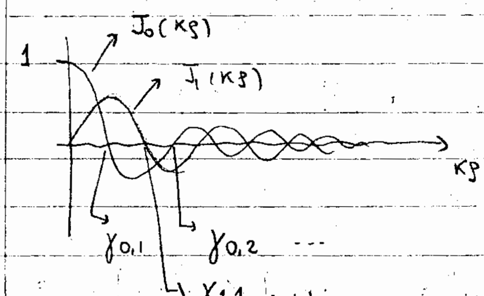
Our solution

$R(\rho) = C_1 J_n(k\rho) + C_2 Y_n(k\rho)$

BC:

$R(0)$ finite $\Rightarrow$ $C_2 = 0$

$R(a) = 0$ $\Rightarrow$ $C_1 \neq 0$ $\boxed{J_n(ka) = 0}$



J with integer index oscillates

zeros of J_n : $\gamma_{n,m}$

$\gamma_{n,m}$: $J_n(\gamma_{n,m}) = 0$ known

$\Rightarrow ka = \gamma_{n,m}$ $\boxed{k = \frac{\gamma_{n,m}}{a}} = \frac{\omega}{c}$

If we want to apply IC's: superposition

$$u = \sum_{\{\omega_{n,m}\}} R_{n,m}(\rho)\, \Theta_n(\theta)\, T_{n,m}(t) = \sum_{n=0}^{+\infty} \sum_{m=1}^{+\infty} R_{nm}(\rho)\, \Theta_n(\theta)\, T_{nm}(t)$$

$$u(\rho,\theta,t) = \sum_{n=0}^{+\infty} \sum_{m=1}^{+\infty} J_n(k_{n,m}\rho) \left[A_{nm} \cos(n\theta) + B_{nm} \sin(n\theta) \right] \cos(\omega_{mn} t)$$

($\sin \omega_{mn} t$ do not apper because $u_t|_{t=0} = 0$)

Find A_{nm} and B_{nm} so that $u(0) = f(\rho,\theta)$

Homogeneous PDE problems:

$$\left\{\begin{array}{l} \Psi_{tt} = \mathcal{L}\Psi \quad (\mathcal{L} \to \text{operator}) \\ \\ + \text{ homogeneous bc's} \end{array}\right. \quad \text{or} \quad \left\{\begin{array}{l} i\Psi_t = \mathcal{L}\Psi \\ \\ + \text{ homogeneous bc's} \end{array}\right.$$

$\mathcal{L}$ only space variables

Find from these problems "Natural frequencies": ω

$$\psi(\vec{r},t) = e^{\pm i\omega t} R(\vec{r})$$ $\Rightarrow$ find ω and R so that the Problem is satisfied

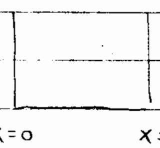
! Applies as well to linear as to non-linear problems

Example: Linear Schrödinger equation: $i\Psi_t = -\Psi_{xx}$, $0 < x < L$ $(V=0)$

"particle in a box" $\Psi(0) = \Psi(L) = 0$

$\omega \leftrightarrow$ "energy levels of the particle"

$\uparrow$ no external forcing



we assume: $e^{-i\omega t}\phi(x)$ (we took the minus because in this case $\omega > 0$ $\to$ minimum of energy)

$$\Rightarrow \left\{\begin{array}{l} \omega\phi = \phi'' \\ \phi(0) = 0, \ \phi(L) = 0 \end{array}\right.$$

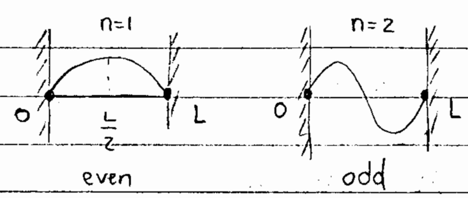
$\Rightarrow$ $\phi(x) = A\sin(\sqrt{\omega}x) + B\cos(\sqrt{\omega}x)$

$\phi(0) = 0 \Rightarrow B = 0$

$\phi(L) = 0 \Rightarrow A\sin(\sqrt{\omega}L) = 0 \Rightarrow \sqrt{\omega}L = n\pi \Rightarrow \boxed{\omega = \left(\frac{n\pi}{L}\right)^2}$ $n = 1, 2, \ldots$

do not dependent on amplitude

Draw the modes:



why the solution preserves this symmetry?

because the operator is symmetric if you replace $x \to -x$, the PDE remains the same.