18.306: Advanced Partial Differential Equations with Applications
M.I.T. Department of Mathematics **Spring 2004**
Practice Set 3 **Handed out**: Wednesday, 03/17/04
Due: NEVER

You are NOT required to return solutions. Some of these problems may be given in Quiz 2. The description of the references given in parentheses can be found in the Bibliography for 18.306.

21. (Carrier & Pearson, Prob. 13.10.3, p. 255.) The equations of the *two-dimensional* isentropic compressible flow (characterized by entropy S=const.) are

$$\rho_t + (\rho u)_x + (\rho v)_y = 0, \quad \rho(u_t + u_x u + u_y v) = -c^2 \rho_x, \quad \rho(v_t + v_x u + v_y v) = -c^2 \rho_y,$$

where $c = c(\rho)$. Discuss the characteristic surfaces in the (x, y, t) space.

22. (Carrier & Pearson, Prob. 13.4.8, p. 239.) The equations of the shallow water theory read as

$$u_t + u u_x + g w_x = 0, \quad w_t + [u(w + d)]_x = 0,$$

where u is the horizontal velocity (taken as independent of vertical position), w is the wave height, d is the depth, and g is the gravity acceleration. The second of these equations is already in the desired form of a conservation law. To interpret these equations as conservation of mass and momentum, obtain another desired conservation law via multiplying the first equation by $(w + d)$, the second by u, and adding. Can you obtain conditions on the possible dicontinuity of u and w (called a *hydraulic jump*)?

23. (Levine, Chap. 24, Prob. 1, pp. 277, 278.) Show that the wave equation with a mixed derivative, $u_{tt} - c^2 u_{xx} = -2\alpha u_{xt}$, where α and c are positive constants, admits the periodic in t solution $u(x, t) = A \sin(\omega t - kx)$. What is the relation between k and ω (called a *dispersion relation*)? Find an approximate formula for $u(x, t)$ for sufficiently small $\alpha x/c$ and α/ω.

24. (a) [Levine, Chap. 4, Prob. 10, p. 49.] Solve the PDE $x^2 u_{xx} + \frac{1}{4} u = u_{tt}$ by seeking particular traveling-wave type solutions of the form $u(x, t) = a(x)\, f(b(x) - t)$, which allow for x-dependent wave speeds; f is an arbitrary function, and a and b satisfy appropriate ODE. (You may wish to take $x > 0$ and $t > 0$.)

(b) [Carrier & Pearson, Prob. 3.4.6, p. 44.] A semi-infinite string in $x > 0$ has a density that varies with x in such a way that the resulting PDE for its displacement $u(x, t)$ is $u_{tt} = c^2 u_{xx}$ where $c^2 = c(x)^2 = (A + Bx)^4$; A and B are known constants. Solve the PDE via the transformation $u = (A + Bx)\psi(x, t)$ followed by the change of variable $\xi = (x/A)(A + Bx)^{-1}$.

(c) [Levine, Chap. 4, Prob. 11, p. 50.] Show that a particular solution of the nonlinear PDE $u_t = [K(u)u_x]_x$ can be obtained in the form $u = f(x - vt)$, which describes a wave form that travels at constant speed v, where f satisfies a first-order ODE.

25. Find the (unbounded in t) temperature distribution $T(x, t)$ for $0 < x < L$ and $t > 0$ given that $T_t = \nu T_{xx}$ and $T(0, t) = 0, T(L, t) = \alpha t$ and $T(x, 0) = 0$.

26. (Exercise on Fourier transforms.) Find a single-integral representation for $u_y(0,0)$ if $u(x,y)$ satisfies the PDE $\nabla^2 u - \lambda^2 u = f(x,y)$ for $-\infty < x < \infty$ and $y > 0$, with $f(x,y) = e^{-ay}\,(b^2 + x^2)^{-1}$, and the conditions $u(x,0) = 0$ and $u(x,\infty) = 0$; a, b and λ are positive constants.

27. (Carrier & Pearson, Prob. 15.4.6, p. 296.) The steady-state ($T_t = 0$) temperature $T(x,y)$ inside the strip ($-\infty < x < \infty$, $0 < y < L$) satisfies the Laplace equation, $T_{xx} + T_{yy} = 0$, with the given conditions $T(x,L) = 0$ and $T(x,0) = f(x)$ (where $f(x) \to 0$ as $|x| \to \infty$). Use the Fourier transform in x in order to find $T(x,y)$. Invert the transform by summing residues, and verify that the result agrees with that obtained by the direct Fourier series expansion method.

28. (Levine, Chap. 35, Prob. 11, p. 596.) In class I said that the Fourier transform (FT) is generally applied in variables that take values in the range $(-\infty, +\infty)$. However, one may also apply FT to a variable in $(0, \infty)$ when the corresponding function is taken to be zero in $(-\infty, 0)$. Consider the following example.

    A low-frequency electromagnetic field in a conducting medium satisfies approximately the diffusion equation. Assume that the (scalar) component $E(x,t)$ of a (vector) electric field in a conducting half space $x > 0$ (say, sea water) satisfies the PDE

$$E_t = (c^2/\sigma) E_{xx}, \quad x > 0,\ t > 0,$$

    along with the conditions $E(0,t) = f(t), E(x,0) = 0$ and $E(x \to \infty, t) \to 0; \sigma$ is the conductivity and c is the speed of light. By applying FT in the time t and contour integration, find an integral representation for $E(x,t)$ in terms of $f(t'),\ 0 < t' < t$, and then derive the energy loss formula

$$Q(t) \equiv \int_0^\infty dx\, \sigma |E(x,t)|^2 = \frac{c}{2}\sqrt{\frac{\sigma}{\pi}} \int_0^t \int_0^t dt'\, dt''\ \frac{f(t')f(t'')}{(2t - t' - t'')^{3/2}}.$$

29. By applying FT, solve the Laplace equation $u_{xx} + u_{yy} = 0$ for $-\infty < x < \infty$ and $y > 0$, subject to the initial conditions (Cauchy data) of the form $u(x,0) = f(x), u_y(x,0) = g(x)$. Is the solution u regular (differentiable) for $y > 0$? Comment on the convergence of the Fourier integral. Argue that a small error in the data can cause a large error in the solution for $y > 0$. (Therefore, you should conclude that the problem is ill-posed in the sense of Hadamard.)

30. (Levine, Chap. 24, Prob. 2, p. 278.) The pendulum-like oscillatory motions of a fluid column inside a U-shaped tube in a vertical plane can be described approximately (in linearized form) by the PDE

$$z_{tt} - \nu(z_{rrt} + r^{-1} z_{rt}) + (2g/L) z = 0, \quad 0 < r < R,\ t > 0,$$

    where $z = z(r,t)$ is the displacement relative to the equilibrium level as a function of the radial distance r from the tube axis and the time $t > 0$, and g, L and ν denote the gravity acceleration, length of the fluid column and kinematic viscocity of the fluid. Describe the appropriate solution $z(r,t)$ subject to the conditions

$$z(r,0) = -H, \quad z_t(r,0) = 0, \quad z_t(R,t) = 0,$$

    of which the last one expresses a vanishing fluid velocity at the tube wall, $r = R$.