18.306 : Advanced PDE with Applications

Handout 5 : Review of 1st-order PDE

I. General Theory for solutions

The main idea in solving a 1st-order PDE is to reduce it to a system of ODE along certain curves called characteristics. For the sake of clarity (and convenience), I review two major cases of PDE.

Ⓐ Quasi-linear 1st-order PDE : $a(x,y,u)\cdot u_x + b(x,y,u)\cdot u_y = c(x,y,u)$ ①

[Of course, this case may include linear 1st-order PDE.]

The characteristics are viewed as curves in the (3-dimensional) (x,y,u) space. The desired system of ODE is written in the form

$$\frac{dx}{a} = \frac{dy}{b} = \frac{du}{c} \quad . \qquad ②$$

In solving this system, you may want to make one of the following choices:

(i) Start with a pair in (2) that finally, i.e., after mutual cancellations, involves only two variables and their differentials and therefore leads to a self-contained ODE.

Example : $x u_x + y\, u_y = 9x^2+y^2$ ③

CHAR : $$\frac{dx}{x} = \frac{dy}{y} = \frac{du}{9x^2+y^2} \quad . \qquad ④$$

Note that it is not a good idea to start with a pair that involves du. For instance, taking $\frac{dx}{x} = \frac{du}{9x^2+y^2}$ will not lead anywhere since you have to deal with all 3 variables (and you need to solve an ODE, with 2 variables!)

A good choice is:

$$\frac{dx}{x} = \frac{dy}{y} \Leftrightarrow d(\ln x) = d(\ln y) \Leftrightarrow y = C_1 x, \quad C_1 = \text{const.}$$

Then, you can replace y by C_1x and determine u as a function of x:

$$\frac{dx}{x} = \frac{du}{9x^2+(C_1x)^2} \Leftrightarrow du = (9x + C_1^2 x)dx \Leftrightarrow u = \frac{9}{2}x^2 + \frac{\overbrace{(C_1x)^2}^{y^2}}{2} + C_2$$

General solution: $u - \frac{9}{2}x^2 - \frac{1}{2}y^2 = F(y/x)$, or $u - \frac{9}{2}x^2 - \frac{1}{2}y^2 = G(\frac{x}{y})$, ⑤

F, G: arbitrary.

— END EXAMPLE —

(ii) Combine terms in (2) so that you can define new variables, as combinations of x, y, u, that enable you to identify and solve a self-contained ODE (involving 2 variables).

Example [Prob. 1(a) of Prac. Set 2]: $(y+u)u_x + (x+u)u_y = x+y$. ⑥

CHAR: $\frac{dx}{y+u} = \frac{dy}{x+u} = \frac{du}{x+y}$. ⑦ Note: No single pair gives a self-contained ODE.

One can make use of the identity $\frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3} = \frac{a_1+a_2+a_3}{b_1+b_2+b_3} = \frac{a_1-a_2}{b_1-b_2} = \frac{a_2-a_3}{b_2-b_3}$

if $\frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3}$.

Hence, ⑦ gives:

$$\frac{d(x+y+u)}{(y+u)+(x+u)+(x+y)} = \frac{d(y-u)}{(x+u)-(x+y)} \Leftrightarrow \frac{d(x+y+u)}{2(x+y+u)} = -\frac{d(u-y)}{u-y}$$

So, treat $x+y+u$ and $u-y$ as new variables:

$$\frac{1}{2}d\ln|x+y+u| = -d\ln|u-y| \Leftrightarrow |x+y+u|^{1/2} = \frac{C_1'}{|u-y|} \Leftrightarrow x+y+u = \frac{C_1}{(u-y)^2}.$$ ⑧

Further, ⑦ gives: $$-\frac{d(y-u)}{y-u} = -\frac{d(x-u)}{x-u} \Leftrightarrow y-u = C_2\cdot(x-u) \qquad ⑨$$

From ⑧ and ⑨, one way to write the general solution is:

$$u = \frac{y-C_2x}{1-C_2} = \frac{y-x\cdot G[(x+y+u)(u-y)^2]}{1-\underset{\llcorner\text{arbitrary}}{G}[(x+y+u)\cdot(u-y)^2]}$$ or, altern., $$(x+y+u)(u-y)^2 = \overset{\text{arbitrary}}{F}\left(\frac{u-x}{u-y}\right).$$

— END EXAMPLE—

An extension of case Ⓐ considers the PDE

$$a(x,y,z,u)\,u_x + b(x,y,z,u)\,u_y + c(x,y,z,u)\,u_z = d(x,y,z,u), \qquad ⑩$$

where the characteristics are viewed as curves in the 4-dimensional (x,y,z,u) space.

CHAR : $$\frac{dx}{a} = \frac{dy}{b} = \frac{dz}{c} = \frac{du}{d}. \qquad ⑪$$

<u>Example</u> : $x u_x + y u_y + u u_z = 0$

CHAR : $$\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{u} = \frac{du}{0} \qquad ⑫$$

$\Rightarrow du = 0 \Rightarrow u = C_1 = \text{const.}$

$\frac{dx}{x} = \frac{dy}{y} \Rightarrow y = C_2x.$

$\frac{dx}{x} = \frac{dz}{\underbrace{u}_{C_1}} \Rightarrow z = C_1\ln x + C_3' \Leftrightarrow x = C_3e^{z/u}$

General solution:

$$u = \underset{\llcorner\text{fcn of 2 variables}}{F}\left(\frac{y}{x},\, xe^{-z/u}\right) \qquad ⑬$$

[Think: u is $C_1 = \text{const.}$ when $\frac{y}{x} = C_2 = \text{const}$ <u>and</u> $xe^{-z/u} = C_3 = \text{const.}$

So, to each (C_2, C_3) there corresponds $u = C_1$ etc].

— END EXAMPLE—

Ⓑ General 1st-order PDE: $H(x,y,u,p,q) = 0$; $p \equiv u_x,\ q \equiv u_y.$ ⑭

Charpit's eqns : $\dfrac{dx}{H_p} = \dfrac{dy}{H_q} = \dfrac{du}{pH_p + qH_q} = \dfrac{-dp}{H_x + H_u p} = \dfrac{-dq}{H_y + H_u q}.$ ⑮

In this case, the characteristics are viewed as curves in the 5-dim. space (x,y,u,p,q).

<u>Example</u> [Prob. 4 of Prac. Set 2] $H = p - q - apq$; $p = u_x$, $q = u_y$. ⑯

Charpit's eqns: $\dfrac{dx}{1-aq} = \dfrac{dy}{-(1+ap)} = \dfrac{du}{p(1-aq) - q(1+ap)} = \dfrac{-dp}{0} = \dfrac{-dq}{0}.$ ⑰

It follows that $p = C_1 = \text{const}$ and $q = C_2 = \text{const}$. along CHAR. ⑱

- $\dfrac{dx}{1-aC_2} = \dfrac{dy}{-(1+aC_1)} \iff -(1+aC_1)x + C_3 = (1-aC_2)\,y$

  $\iff (1-aq)y = -(1+ap)x + C_3.$ ⑲

- $\dfrac{dx}{1-aC_2} = \dfrac{du}{C_1 - C_2 - 2aC_1C_2} \iff (1-aC_2)u = (C_1 - C_2 - 2aC_1C_2)\,x + C_4$

  $\iff (1-aq)u = (p - q - 2apq)x + C_4$ ⑳

The general solution to ⑯ is described (in terms of arbitrary constants C_1, C_2, C_3, C_4) by Eqs ⑱ - ⑳; this form is called a complete integral of. ⑯.

If initial data is given for u, we can determine C_1, C_2, C_3 and C_4 in terms of data parameters and then find the solution to ⑯ either explicitly or in an implicit form $f(x,y,u) = 0$. —END EXAMPLE—

Example : Solve $H = p - q - apq = 0$ with $u(x,0) = x$.

Parametrize data $(x=s)$: $u(s,0) = s \Rightarrow p = 1$.

$$H=0 \Rightarrow q = \frac{p}{1+ap} = \frac{1}{1+a}.$$

From p. 4, we express C_1, C_2, C_3 and C_4 in terms of s:

$$C_1 = 1, \quad C_2 = \frac{1}{1+a}. \qquad ㉑$$

⑲ : $\left(1 - \frac{a}{1+a}\right)\cdot 0 = -(1+a)s + C_3 \Rightarrow C_3 = (1+a)s$ ㉒

⑳ : $\left(1 - \frac{a}{1+a}\right)\cdot s = \left(1 - \frac{1}{1+a} - 2a\frac{1}{1+a}\right)s + C_4 \Rightarrow C_4 = s.$

So, ⑲ : $\frac{1}{1+a}y = -(1+a)x + (1+a)s \Leftrightarrow s = x + \frac{y}{(1+a)^2}$. ㉓

⑳ : $\frac{u}{1+a} = \left(1 - \frac{1}{1+a} - \frac{2a}{1+a}\right)x + s \Leftrightarrow$

$$\overset{㉓}{\Leftrightarrow} \quad u(x,y) = x + \frac{y}{1+a}. \qquad ㉔$$

- END EXAMPLE -

II. Shocks and shock conditions

- In the particular case where the PDE takes the form of a conservation law,

$$\rho_t + q_x = 0, \qquad \rho\text{: "density"}, \quad q\text{: "flux"}, \qquad ㉕$$

we showed in class that $\rho=\rho(x,t)$ may become multiple-valued. This situation is remedied, within ㉕, by allowing for a discontinuous solution, called "shock." The "shock condition" on the shock speed $\mathcal{U} = \dot{x}_{sh}(t)$, if a shock develops, is

$$\mathcal{U} = \frac{q^+ - q^-}{\rho^+ - \rho^-} \qquad ㉖$$

{important; remember this!}

where $q^{\pm}, \rho^{\pm}$ are the values of q and ρ left $(-)$ and right $(+)$ of the shock.

[Recall: In order to derive ㉖ we used the integrated form of ㉕, viz.,

$$\frac{d}{dt}\int_{x_2}^{x_1} dx\ \rho(x,t) = q(x_2,t) - q(x_1,t) \quad , \qquad \text{where} \quad x_2 < x_{sh} < x_1 \ ;$$

so no explicit assumption has to be made about the specific form of $q(x,t)$.]

- More particularly, when $q = Q(\rho)$ (as in traffic flow), (26) reads

$$\mathcal{U} = \frac{Q(\rho^+) - Q(\rho^-)}{\rho^+ - \rho^-} \ . \qquad ㉗$$

By defining $F(\xi) \equiv c\big(\overset{\xi \text{ parametrizes the initial data!}}{\rho(\xi,0)}\big)$ where $c(\rho) \equiv Q'(\rho)$ and $\rho(x,0)$ is the initial density (at $t=0$), $\rho(x,t)$ starts to develop a shock when

$$t = t_B = \frac{1}{|F'(\xi)|_{max}} \qquad \text{(earliest breaking time).} \qquad ㉘$$

The condition for a shock to occur is

$$\boxed{F'(\xi) < 0} \ \text{somewhere.} \qquad ㉙$$

Then, ρ_x and ρ_t become infinite when

$$\rho_x, \rho_t \to \infty \quad : \qquad \underline{1 + F'(\xi)\cdot t = 0} \quad , \qquad (30)$$

where ξ parametrizes the value $\rho(\xi,0)$ of the initial data corresponding to this time t.

Remark: Eq. (28) may even account for the case of discontinuous $\rho(x,0)$; just consider discont. $\rho(x,0)$ as the limit of a smooth $\rho(x,0)$ with the slope $|F'(\xi)|$ becoming infinitely large! $\Rightarrow t_B \to 0^+$.

III. PDE systems

With $u_j = u_j(x,t)$ and $j = 1,2,\ldots,n$, an $n \times n$ system of PDE for u_j is

$$A_{ij}\, u_{j,t} + a_{ij}\, u_{j,x} + b_i = 0, \qquad i,j = 1,2,\ldots,n, \qquad (31)$$

where $\underbrace{A_{ij},\ a_{ij}}_{\text{matrices}}$ and $\underbrace{b_i}_{\text{vector}}$ may depend on u_j, x, t but not on $u_{j,x}$ and $u_{j,t}$.

To find the characteristics of (31) satisfying

$$\delta x = \alpha \cdot \varepsilon, \quad \delta t = \beta \cdot \varepsilon, \qquad (32)$$

i.e., in order to determine the characteristic directions (α,β), it suffices to find (α,β) so that the system of linear equations

$$\boxed{\ell_i\,(\alpha A_{ij} - \beta\, a_{ij}) = 0} \qquad (33)$$

has non-trivial (non-zero) solutions. Hence,

$$\boxed{\det[\alpha A_{ij} - \beta\, a_{ij}] = 0.} \qquad (34)$$

The system (31) is called hyperbolic if (33) is solved by n linearly independent vectors $\underline{\ell}^{(k)} = (\ell_1^{(k)}, \ell_2^{(k)}, \ldots, \ell_n^{(k)})$, where $k = 1,2,\ldots,n$, and the corresponding $(\alpha^{(k)}, \beta^{(k)})$ are real with $\alpha^{(k)2} + \beta^{(k)2} \neq 0$.

[If there exist non-real (complex) solutions (α,β) of (34), then the system (31) is called elliptic. More about this in class...].

If we assume that $\beta \neq 0$, (34) gives

$$\det[a_{ij} - \kappa A_{ij}] = 0, \qquad (34')$$

where $\kappa \equiv \dfrac{\alpha}{\beta}$. In the case where t is time, the ratio $\kappa = \dfrac{\alpha}{\beta} = \dfrac{\delta x}{\delta t} \to \dfrac{dx}{dt}$ is denoted by $c = \dfrac{dx}{dt}$: characteristic speed of system (31).

Problems - Examples

$\boxed{1}$ (P.6 of P.Set 1) $\quad yu_x - xu_y = 0, \quad x = \rho\cos\varphi, \quad y = \rho\sin\varphi.$

Hint: Express $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}$ in terms of ρ, φ; $\rho = \sqrt{x^2+y^2}, \quad \varphi = \tan^{-1}(y/x)$. :

$$\frac{\partial}{\partial x} = \frac{\partial\rho}{\partial x}\frac{\partial}{\partial\rho} + \frac{\partial\varphi}{\partial x}\frac{\partial}{\partial\varphi} = \frac{x}{\rho}\frac{\partial}{\partial\rho} + \left(-\frac{y}{x^2}\cdot\frac{x^2}{\rho^2}\right)\frac{\partial}{\partial\varphi} = \cos\varphi\frac{\partial}{\partial\rho} - \sin\varphi\frac{\partial}{\partial\varphi} ;$$

$$\frac{\partial}{\partial y} = \frac{\partial\rho}{\partial y}\frac{\partial}{\partial\rho} + \frac{\partial\varphi}{\partial y}\frac{\partial}{\partial\varphi} = \frac{y}{\rho}\frac{\partial}{\partial\rho} + \left(\frac{1}{x}\;\frac{x^2}{\rho^2}\right)\frac{\partial}{\partial\varphi} = \sin\varphi\frac{\partial}{\partial\rho} + \cos\varphi\frac{\partial}{\partial\varphi} .$$

PDE in ρ, φ: $\quad \cancel{\rho}\sin\varphi\left[\underbrace{\cos\varphi\frac{\partial u}{\partial\rho} - \sin\varphi\frac{\partial u}{\partial\varphi}}_{u_x}\right] - \cancel{\rho}\cos\varphi\left[\underbrace{\sin\varphi\frac{\partial u}{\partial\rho} + \cos\varphi\frac{\partial u}{\partial\varphi}}_{u_y}\right] = 0$

$$\Longleftrightarrow \quad -(\sin^2\varphi + \cos^2\varphi)\frac{\partial u}{\partial\varphi} = 0 \quad \Longleftrightarrow \quad \frac{\partial u}{\partial\varphi} = 0 \quad !$$

$\boxed{2}$ (P. 8 of P.Set 1) $\quad yu_x + xu_y = 2xy.$

(a) CHAR: $$\frac{dx}{y} \overset{①}{=} \frac{dy}{x} \underset{②}{=} \frac{du}{2xy}$$

①: $\quad x\,dx = y\,dy \iff x^2 - y^2 = C_1 = \text{const.}$

②: $\quad dy = \frac{du}{2y} \iff u = y^2 + C_2$

$\Big\}\quad u(x,y) = y^2 + \overset{\text{arbitrary}}{F}(x^2-y^2)$ (general solution)

(b) $\quad u = y \quad$ on $\quad x^2+y^2 = 1 \Rightarrow y^2 = 1-x^2$, or $x^2 = 1-y^2$.

From general solution: $\quad y = y^2 + F(1-2y^2) \iff F(1-2y^2) = y - y^2.$

Set $\quad W = 1-2y^2 \Rightarrow y = \pm\sqrt{\frac{1}{2}(1-W)} \quad : \quad F(w) = \underbrace{\pm\sqrt{\frac{1}{2}(1-w)} - \frac{1}{2}(1-w)}_{\text{I please check algebra!}}.$

~~Put~~ back to general solution (and replace w by x^2-y^2 !):

$$u(x,y) = y^2 + \left[\pm \sqrt{\tfrac{1}{2}(1-x^2+y^2)} - \tfrac{1}{2}(1-x^2+y^2) \right] \quad \text{etc.}$$

This solution representation is valid for $1-x^2+y^2>0 \Leftrightarrow x^2-y^2<1$. (bec. of sq. root)

A better way to write this solution stems from eliminating the square root:

$$u = y^2 - \frac{1}{2} + \frac{x^2}{2} - \frac{y^2}{2} \pm \sqrt{\tfrac{1}{2}(1-x^2+y^2)} \Leftrightarrow u - \frac{x^2}{2} - \frac{y^2}{2} + \frac{1}{2} = \pm \sqrt{\tfrac{1}{2}(1-x^2+y^2)}$$

$$\Leftrightarrow \boxed{\left(u - \frac{x^2+y^2-1}{2}\right)^2 = \frac{1-x^2+y^2}{2}}$$, which represents a surface in (x,y,u).

(no restriction is posed on x,y)

3 (P. 10 of Pr. Set 1) $u_x + u_y = u^2$

CHAR: $$\overbrace{\underbrace{\frac{dx}{1} = \frac{dy}{1}}_{①} = \frac{du}{u^2}}^{②}$$

①: $y - x = C_1 = \text{const.}$ ②: $d\left(-\frac{1}{u}\right) = dx \Leftrightarrow \frac{1}{u} = -x + C_2$

Hence, $u = \dfrac{1}{F(y-x) - x}$: general solution ; F: arbitrary

$u = x$ on $y = -x$: $\overbrace{-y}^{u} = \dfrac{1}{F(\cancel{-}2y) + y} \Leftrightarrow F(2y) + y = \dfrac{-1}{y} \Leftrightarrow F(2y) = \dfrac{-1}{y} - y$

Thus; $u = \dfrac{1}{\frac{-2}{y-x} - \frac{y-x}{2} - x} = \dfrac{2(x-y)}{4 + (x-y)^2 - 2x(x-y)}$.

$\boxed{\text{Where does } u \to \infty \text{ ?}}$

For what (x,y)

$\boxed{4}$ (P. 6 of Pr. Set 2) $\boxed{\text{Hint:}}$

By definition of μ, if we multiply the ODE by μ,

$$\mu a\, dx + \mu b\, dy = 0 \quad ,$$

the ODE should be in the form

$$P dx + Q dy = 0 \quad ; \qquad P = \mu a, \; Q = \mu b ,$$

where the LHS should be the exact differential of some R :

$$dR = P dx + Q dy = 0 ,$$

i.e., $P = \dfrac{\partial R}{\partial x}$ and $Q = \dfrac{\partial R}{\partial y}$. Therefore, the condition on μ is

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \quad \Leftrightarrow \quad \frac{\partial(\mu a)}{\partial y} = \frac{\partial(\mu b)}{\partial x} .$$

PDE for μ : $\quad b\mu_x - a\mu_y = a_y \mu - b_x \mu$

CHAR : $\quad \dfrac{dx}{b} = \dfrac{dy}{-a} = \dfrac{d\mu}{\mu(a_y - b_x)}$.

For $a = 3xy + 2y^2$, $b = 3xy + 2x^2 \Rightarrow \quad a_y - b_x = (3x + 4y) - (3y + 4x) = -x + y$.

$$\Rightarrow \quad \frac{dx}{3xy + 2x^2} = \frac{dy}{-(3xy + 2y^2)} = \frac{d\mu}{-\mu(x-y)}$$

From first two equations: $\quad \underbrace{\dfrac{d(x+y)}{2(x^2-y^2)} = \dfrac{dx}{3xy+2x^2} = \dfrac{dy}{-(3xy+2y^2)} = \dfrac{d\mu}{-\mu(x-y)}}_{①}$

① : $\quad \dfrac{d(x+y)}{2\cancel{(x-y)}(x+y)} = \dfrac{d\mu}{-\mu\,\cancel{(x-y)}} \quad \Leftrightarrow \quad d\ln|x+y| = -2\, d\ln|\mu| \quad \Leftrightarrow \quad \mu = \dfrac{C_1}{(x+y)^{1/2}}$

(new variable: x+y)

For $C_1 = 1$ we get $\boxed{\mu = \dfrac{1}{(x+y)^{1/2}}}$ ... (CONTINUE!)

$\boxed{5}$ (Problem from gas dynamis; similar to Prob. 10 Prac. Set 2)

The density ρ, velocity u, entropy S and pressure p of a gas satisfy the PDE system

$$\begin{cases} \rho_t + u\rho_x + \rho u_x = 0 \\ u_t + uu_x + \frac{1}{\rho} p_x = 0 \\ S_t + u S_x = 0 \end{cases}$$

where $p = p(\rho, S)$ must be eliminated through: $\left\{ p_x = \rho_x \underbrace{\left(\frac{\partial p}{\partial \rho}\right)}_{\equiv a^2} + S_x \underbrace{\left(\frac{\partial p}{\partial S}\right)}_{\equiv b^2} \right.$

$\llcorner \frac{\partial p}{\partial \rho} > 0$ (by assumption)

Hence, the system takes the matrix form

$$\underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{[A_{ij}]} \cdot \begin{pmatrix} \rho_t \\ u_t \\ S_t \end{pmatrix} + \underbrace{\begin{pmatrix} u & \rho & 0 \\ a^2/\rho & u & b^2/\rho \\ 0 & 0 & u \end{pmatrix}}_{[a_{ij}]} \begin{pmatrix} \rho_x \\ u_x \\ S_x \end{pmatrix} = 0 .$$

The <u>characteristic speeds</u> $c = \frac{dx}{dt}$ are found via solving

$$\det[a - cA] = 0 \iff \begin{vmatrix} u-c & \rho & 0 \\ a^2/\rho & u-c & b^2/\rho \\ 0 & 0 & u-c \end{vmatrix} = 0 \iff (u-c)\cdot[(u-c)^2 - a^2] = 0.$$

$$\Longrightarrow \begin{cases} c = u \\ c = u \pm a \end{cases} \quad \left(\text{taking } a > 0, \text{ with } \frac{\partial p}{\partial \rho} \equiv a^2 > 0 \right).$$

To find the characteristic equations, i.e., the combinations of variations of ρ, u, S that remain 0 along the characteristics in this case, one must solve for l_i:

$$l_i \cdot (a_{ij} - cA_{ij}) = 0 \quad \text{for} \quad c = u, \quad c = u \pm a$$

Answers (see also Ex. 8, p. 121 of Whitham; only answers are given there too):

(i) $\frac{dx}{dt} = c = u$: $\quad (l_1 \ l_2 \ l_3) \begin{pmatrix} 0 & \rho & 0 \\ a^2/\rho & 0 & b^2/\rho \\ 0 & 0 & 0 \end{pmatrix} = 0 \Rightarrow \begin{cases} l_1 = 0 \\ l_2 = 0 \\ l_3 : \text{arbitrary} \neq 0 \end{cases}$

Combination of eqns: $\quad 0 \cdot (\rho_t + u\rho_x + \rho u_x) + 0 \cdot (u_t + uu_x + \frac{1}{\rho} p_x) + l_3 \cdot (S_t + uS_x) = 0$

$$\Rightarrow \quad S_t + \underbrace{u}_{\frac{dx}{dt}} S_x = 0 \Rightarrow \quad \frac{dS}{dt} = 0 \quad \text{along} \quad \text{CHAR} \quad \frac{dx}{dt} = u.$$

(ii) $\frac{dx}{dt} = u + a$: $\quad (l_1 \, l_2 \, l_3) \cdot \begin{pmatrix} -a & \rho & 0 \\ a^2/\rho & -a & b^2/\rho \\ 0 & 0 & -a \end{pmatrix} = 0 \Rightarrow \begin{cases} l_1 = a^2 \swarrow \text{set} \\ \rightarrow l_2 = a\rho \\ \rightarrow l_3 = b^2 \end{cases}$

It follows that $\quad \frac{dp}{dt} + a\rho \frac{du}{dt} = 0 \quad$ along $\quad \frac{dx}{dt} = u + a$

(iii) $\frac{dx}{dt} = u - a$ $\quad (a \leftrightarrow -a)$: $\quad \frac{dp}{dt}$ [illegible] $a\rho \frac{du}{dt} = 0$ along CHAR.

Is the system hyperbolic or not? Explain.