April 12, 2004 | Lecture 18

Opt. Reading : WHI 11.1- 11.6

Extra Lecture : FRI , hrs TBA

Review| Green's function

Ex.1 Poisson eqn. $\begin{cases} \nabla^2 \phi = -\rho(\vec{r}, \dots) \\ \phi \to 0 , \quad |\vec{r}| \to \infty \end{cases}$

Green fcn G : $\begin{cases} \nabla^2 G(\vec{r}, \vec{r}') = \overbrace{-\delta(\vec{r} - \vec{r}')}^{\text{point charge}} \\ G \to 0 , \quad |\vec{r}| \to \infty \end{cases}$

$(\rho \to \delta)$

If ρ depends on ϕ

$\Rightarrow$ Solution $\phi(\vec{r}) = \underset{\text{all space}}{\iiint} d\vec{r}\,' G(\vec{r}, \vec{r}\,') \rho(\vec{r}\,', \dots)$ $\Rightarrow$ integral equation

Wave eq. in non-linear medium

Ex.2 $\begin{cases} u_{tt} - c^2 u_{xx} = \underbrace{\boxed{-\mu u_{xx} u_x^2}}_{} \equiv \rho(x,t) & -\infty < x < \infty , \ t > 0 \\ u(x,0) = a(x) , \quad u_t|_{t=0} = b(x) \end{cases}$

Want to write $u(x,t) = u_h(x,t) + u_p(x,t)$

$\rho = 0$

$\begin{cases} u_{h,tt} - c^2 u_{h,xx} = 0 \\ u_h(x,0) = a(x) , \quad u_{h,t}|_{t=0} = b(x) \end{cases}$

$u_p(x,t) = \int_0^\infty dt' \int_{-\infty}^{+\infty} dx' G(x,t; x',t') \rho(x',t')$

satisfies the usual wave eq.

+ Cauchy data

homogeneous BC

* We applied supperposition independently of the fact that the equation is non-linear.

* How to define G?

Define: G replace $g \to \delta$ $\quad \begin{cases} G_{tt} - c^2 G_{xx} = \delta(x-x')\delta(t-t') \\ + \text{"conditions"} \end{cases}$

Verify that $u = u_h + u_p$ satisfies the PDE

$$u_{tt} - c^2 u_{xx} = \underbrace{(u_{h,tt} - c^2 u_{h,xx})}_{0} + \underbrace{(u_{p,tt} - c^2 u_{p,xx})}_{\int_0^\infty dt' \int_0^\infty dx' \underbrace{[\delta(x-x')\delta(t-t')]}_{\text{from } G_{tt} - c^2 G_{xx}} g(x',t')} = g(x,t) \equiv -\mu u_{xx} u_x^2$$

Back to finding G:

$$\begin{cases} G_{tt} - c^2 G_{xx} = \delta(x-x')\,\delta(t-t') \,, & -\infty < x,t < \infty \\ G(x,0) = 0 \\ G_t(x,0) = 0 \end{cases}$$

Note: $\begin{cases} \text{The problem is defined for } t>0 \\ \text{or defined for all } t \text{ and } = 0 \text{ for } t<0 \end{cases}$

→ the BC can be replaced by $G = 0$ for $t<0$

Linear PDE defined for all x, t and translation invariant
→ apply Fourier Transform

FT in x, t: $$G(x,t;x',t') = \int_{-\infty}^{+\infty} \frac{dk_1}{2\pi} \int_{-\infty}^{+\infty} \frac{dk_0}{2\pi}\, e^{\underbrace{ik_1x - ik_0t}_{\text{convention}}}\, \tilde{G}(k_1, k_0; x', t')$$

when you have both space and time, put different signs

$$\tilde{G}(k_1, k_0; x', t') = \int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} dt\, e^{-ik_1x + ik_0t}\, G(x,t;x',t') = FT\{G\}$$

$FT\{G_{tt}\} = -k_0^2 \tilde{G}$, $FT\{G_{xx}\} = -k_1^2 \tilde{G}$

(condition: G has to go to infinity as $t \to \infty$ and $G_t \to 0$ as $t \to \infty$)

(condition: same but with x)

PDE: $-k_0^2 \tilde{G} + c^2 k_1^2 \tilde{G} = FT\{\delta(x-x')\,\delta(t-t')\} = e^{-ik_1x' + ik_0t'}$

$\Rightarrow \tilde{G} = \dfrac{e^{-ik_1x' + ik_0t'}}{c^2k_1^2 - k_0^2}$

$\Rightarrow G(x,t;x',t') = \displaystyle\int_{-\infty}^{+\infty} \frac{dk_1}{2\pi} \int_{-\infty}^{+\infty} \frac{dk_0}{2\pi} \; \frac{e^{ik_1(x-x') - ik_0(t-t')}}{c^2k_1^2 - k_0^2}$

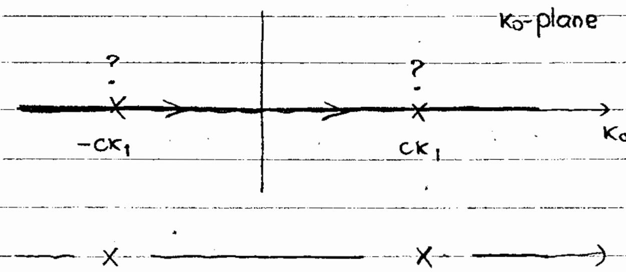
If we take real path of integration denominator becomes 0: $c^2k_1^2 - k_0^2 = 0$ $\quad ck_1 = \pm k_0$

! we can not integrate through these points

Let's integrate first in time / k_0:

$G = \displaystyle\int_{-\infty}^{+\infty} \frac{dk_1}{\pi 2}\, e^{ik_1(x-x')} \underbrace{\int_{-\infty}^{+\infty} \frac{dk_0}{2\pi}\, \frac{e^{-ik_0(t-t')}}{c^2k_1^2 - k_0^2}}_{I(k_1)}$

for real k_1 (fixe) there are points where the integrand of $I(k_1)$ blows up.

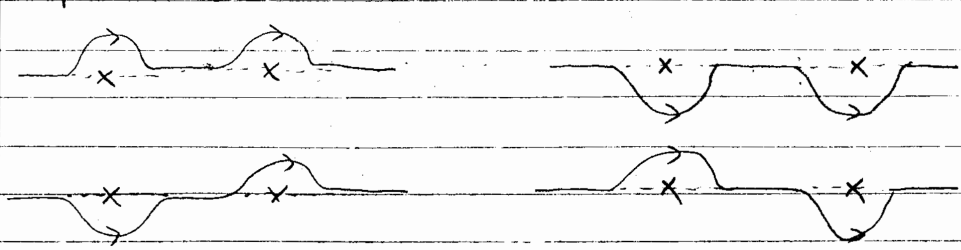


$(k_1 > 0)$

the path should not go through these two points!

Fourier integral is fine as long as the end points of the path are at $\mp\infty$; doesn't matter how do we go in the middle.

4 possibilities:



Digression: $\int_\Gamma dz\, f(z)\, e^{i\lambda z}$ ← real

f(z) has the properties:

(i) $f(z_j) = \infty$ blows up at some points in the complex plain

(ii) $(z-z_j)\, f(z)$: analytic in a neighborhood of each z_j

(i.e z_j is simple pole)

Then,

$$\int_\Gamma dz\, f(z)\, e^{i\lambda z} = \begin{cases} \lambda>0 : \; 2\pi i \sum\limits_{\{z_j\}\text{ above }\Gamma} e^{i\lambda z_j}\, c_j \\ \lambda<0 : \; -2\pi i \sum\limits_{\{z_j\}\text{ below }\Gamma} e^{i\lambda z_j}\, c_j \end{cases} \quad ; \quad c_j = \lim_{z\to z_j}\left[(z-z_j)\, f(z)\right]$$

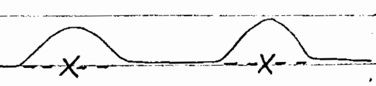
How to choose the path:

→ look at the conditions: $G=0$, $t<0$, $G_t=0$, $t<0$

$$I(\kappa_1) = \int_\Gamma \frac{d\kappa_0}{2\pi}\, e^{-i\kappa_0(t-t')}\, \frac{1}{c^2\kappa_1^2-\kappa_0^2}, \quad \lambda = -(t-t') \begin{cases} <0 & t>t' \\ >0 & t<t' \end{cases} (t'>0)$$

$t<0$ and $t'>0$ means $t<0<t_1$ $\quad \lambda = -(t-t')>0$

Only acceptable path: $\boxed{\lambda>0}$

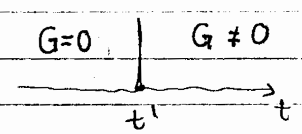


$$I(\kappa_1) = \int_\Gamma \frac{d\kappa_0}{2\pi}\, e^{-i\kappa_0(t-t_1')}\, \frac{1}{c^2\kappa_1^2-\kappa_0^2} = 0, \quad t<t_1$$

Other case $\lambda<0$

$$I(\kappa_1) = \frac{\sin[\kappa_1(t-t')]}{\kappa_1}, \quad t>t' \qquad \text{(Exercise)}$$

Final solution:



$$G(x,t;x',t') = 0, \quad t<t'$$

└ "classical causality": the Green function does not exist before the moment it was excited

$$G(x,t;x',t') = \int_{-\infty}^{+\infty} \frac{d\kappa_1}{2\pi}\, e^{i\kappa_1(x-x')} \cdot \frac{\sin[\kappa_1(t-t')]}{\kappa_1}, \quad t>t'$$

troublesome term is $\frac{1}{\kappa_1}$ $\Rightarrow$ diff. $\frac{\partial}{\partial t}$ in order to eliminate it

$$\frac{\partial G}{\partial t} = \int_{-\infty}^{+\infty} \frac{d\kappa_1}{2\pi}\, e^{i\kappa_1(x-x')}\, \underbrace{\cos[\kappa_1(t-t')]}_{\frac{e^{i\kappa_1(t-t')} + e^{-i\kappa_1(t-t')}}{2}}$$

$$\frac{\partial G}{\partial t} = \int_{-\infty}^{+\infty} \frac{d\kappa_1}{4\pi}\, e^{i[\kappa_1(x-x') + \kappa_1(t-t')]} + e^{i[\kappa_1(x-x') - \kappa_1(t-t')]}$$

Recall: $$\delta(\xi) = \int_{-\infty}^{+\infty} \frac{dk}{2\pi}\, e^{\pm ik\xi}$$

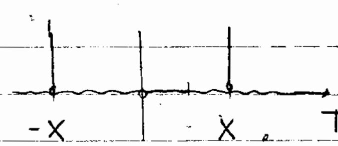
for $t > t'$

$$\frac{\partial G}{\partial t} = \frac{1}{2}\left[\delta(\underbrace{x-x'}_{X} + \underbrace{t-t'}_{T}) + \delta(x-x'-(t-t'))\right]$$

$$\Rightarrow G(x,t;x',t') = \int_{t'}^{t} dt''\, \frac{\partial G}{\partial t''} \underset{T''=t''-t'}{=} \int_{0}^{T} dT''\, \frac{\partial G}{\partial T''}$$

$\downarrow$ because $G=0$ for $t<t'$

$$= \frac{1}{2}\int_{0}^{T} dT''\left[\delta(X+T'') + \delta(X-T'')\right] = \frac{1}{2}\begin{cases} 1, & |T| > |X| \\ 0, & |T| < |X| \end{cases}$$



(zero or one δ are taken into account)

$u_h(x,t)$ = D'Alembert solution (last time)

$$u_p(x,t) = \int_{0}^{+\infty} dt' \int_{-\infty}^{+\infty} dx'\, \frac{1}{2}\,\theta(|t-t'| - |x-x'|)$$

(G non zero if $|t-t'| > |x-x'|$)

(here $t' < t$, otherwise $G = 0$)

$$\boxed{u_p(x,t) = \int_{0}^{t} dt' \int_{x-(t-t')}^{x+(t-t')} dx'\, \frac{1}{2}(-\mu)\, u_{xx}(x',t')\, u_x^2(x',t')}$$

We wrote the particular solution $u_p(x,t)$ as an integral of the unknown solution u. Why is this useful?

$$u(x,t) = u_h(x,t) - \frac{\mu}{2}\int_0^t dt' \int_{x-(t-t')}^{x+(t-t')} dx' \, u_{xx}(x',t')\, u_x^2(x',t')$$

INTEGRAL EQ.

→ contains all the information we know about u in one equation (PDE + conditions)

→ better for numerical treatment

→ analytically : we can make approximations

Suppose non-linearity is small : $\mu \ll 1$

(i) zero order solution $\mu = 0$

$$u(x,t) \approx u_h(x,t) = u^{(0)}(x,t)$$

(ii) next step $\mu \neq 0$

Solve by ITERATION :

$$u(x,t) \approx u^{(1)}(x,t) = u_h(x,t) - \frac{\mu}{2}\int_0^t dt' \int_{x-(t-t')}^{x+(t-t')} dx' \, u^{(0)}_{xx}(x',t')\, u^{(0)2}_x(x',t')$$

1st order approximation for u

and so on...

→ The Green's function can be used to get some insight for non-linear PDE's.

There are restrictrictions :

- here we did REGULAR PERTURBATION
  if you set small parameter = 0 you can still find a solution that satisfies all conditions
  (G solves a linear problem, does not depend on μ)

- SINGULAR PERTURBATIONS
  (modify G so it depends on μ)