April 16, 2004 Lecture 20

Pick up: cookies, Solutions to Hmwk 4

START on Hmwk 5!

Dispersive waves (Review)

- Linear PDE $\Rightarrow \underset{\text{Try wave sol.}}{u = e^{ikx - i\omega t}} \Rightarrow \omega = W(k)$ DISPERSION RELATION

- Superposition: $I(x,t) = \int_{-\infty}^{+\infty} \frac{dk}{2\pi} e^{\overbrace{ikx - iW(k)t}^{\equiv -i\phi(k)}} F(k)$ ← from data

$$\left.\begin{array}{l} x \to \infty \\ t \to \infty \end{array}\right\} \frac{x}{t} \text{ fixed} \Rightarrow I(x,t) \approx \frac{F(\bar{k})}{\sqrt{2\pi |t W''(\bar{k})|}} e^{-i\phi(\bar{k}) - i\frac{\pi}{4}\operatorname{sgn}(W''(\bar{k})t)}$$

$\hookrightarrow \bar{k}: \; W'(\bar{k}) = \frac{x}{t}$

Result by METHOD OF STATIONARY PHASE
METHOD OF STEEPEST DESCENT (two faces of the same coin)

Remarks:

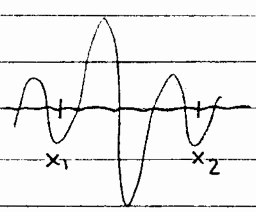
(i) For given $(x,t) \Rightarrow \bar{k} = \bar{k}(x,t)$ (everything is viewed as function of (x,t))
the amplitude / phase change with x, time $\Rightarrow$ Non-uniform wave

(ii) An observer who wants to see the same local wave number $\bar{k}$, and local frequency, $W(\bar{k})$ needs to move with speed $W'(\bar{k})$
This is called group velocity.

$\frac{dW(k)}{dk}$: group velocity at k

different from

$\frac{W(k)}{k}$: phase velocity (an observer moving at c_{ph} sees the same phase / same crest)



Energy between x_1 and x_2:

$$Q(t) = \int_{x_1}^{x_2} dx \, (\text{amplitude})^2$$

$$\approx \int_{x_1}^{x_2} dx \frac{|F(\bar{k}(x,t))|^2}{2\pi t |W''(\bar{k})|}$$

assume $W''(\bar{k}) > 0$ without lost of generality, otherwise we have to play with signs

Change variable: $x = W'(\bar{k}) t \Rightarrow dx = W''(\bar{k}) \cdot t \, d\bar{k}$

$\hookrightarrow \bar{k}(x,t)$ dependence of x is inside

$$\Rightarrow Q(t) = \int_{\bar{\kappa}_1}^{\bar{\kappa}_2} \frac{d\bar{\kappa}}{2\pi}\, W''(\bar{\kappa})\, t\, \frac{|F(\bar{\kappa})|^2}{t\, W''(\bar{\kappa})} = \int_{\bar{\kappa}_1}^{\bar{\kappa}_2} \frac{d\kappa}{2\pi} |F(\bar{\kappa})|^2$$

consistent with Fourier integrals:
integral over x space = integral over κ space

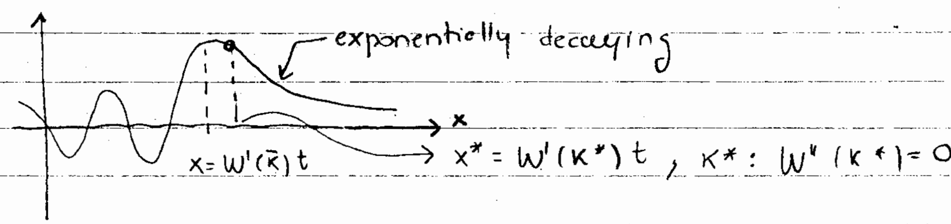
(but we used an approximation here
this does not have to be the case)

Suppose I want
$Q = \text{const} \iff \begin{cases} \bar{\kappa}_1 = \text{const} \\ \bar{\kappa}_2 = \text{const} \end{cases} \Leftarrow \begin{cases} x_1 = x_1(t) \\ x_2 = x_2(t) \end{cases}$ must move with corresponding group velocity

Where the formula for $I(x,t)$ fails?
- we assumed $W''(\bar{\kappa}) \neq 0$
  the formula for $I(x,t)$ breaks down at $\bar{\kappa} : W''(\bar{\kappa}) = 0$

In "real life", we observe

t: fixed



Assume there is $\kappa^* : W''(\kappa^*) = 0$ (independent of x, t comes from the disp. rel.)

define ↓

$W'(\kappa^*) = \frac{x}{t}$ defines x/t

$\underbrace{}$
LINE: CAUSTIC

Remark: None of these assumptions interfere with the genercality of the conclusions we will reach.

Problem: $\bar{\kappa} \to \kappa^*$? (what happens if these are very close)

Assumptions: (Water waves): $\begin{cases} W(\kappa): \text{odd fcn of } \kappa \Rightarrow W(-\kappa) = -W(\kappa) \\ \cancel{F(\kappa): \text{even fcn of } \kappa \Rightarrow F(-\kappa) = F(\kappa)} \end{cases}$

and $\kappa^* = 0$

$$I(x,t) = \int_{-\infty}^{+\infty} \frac{dk}{2\pi} F(k)\, e^{ikx - iW(k)t}$$

$$= \int_0^{\infty} \frac{dk}{2\pi} F(k)\, e^{ikx - iW(k)t} + \underbrace{\int_{-\infty}^{0} \frac{dk}{2\pi} F(k)\, e^{ikx - iW(k)t}}_{\int_0^{\infty} \frac{dk}{2\pi} F(k)\, e^{-ikx + iW(k)t}}$$

$$= \int_0^{\infty} \frac{dk}{\pi} F(k) \cos(kx - W(k)t)$$

Same limit as before:

$\left.\begin{matrix} x \to \infty \\ t \to \infty \end{matrix}\right\} \frac{x}{t}$ fixed

Approximate $W(k)$ around $k^* = 0$

$$W(k) \approx \overset{0 \text{ since } W \text{ is odd}}{W(0)} + k\, W'(k^*)_{\to 0} + \cancel{\frac{1}{2} k^2 W''(k^*)_{\to 0}}^{0} + \frac{1}{3!} k^3 W'''(k^*)_{\to 0}$$

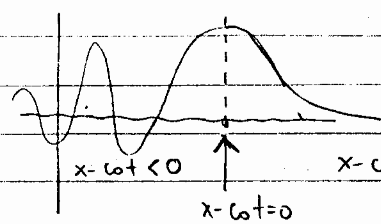
$$I(x,t) \approx F(k^*) \int_0^{\infty} \frac{dk}{\pi} \cos(\quad kx - (c_0 k - \gamma k^3)t)$$

where $c_0 = W'(0)$, $\gamma = -\frac{1}{3!} W'''(0)$

$$I(x,t) \approx F(k^*) \int_0^{\infty} \frac{dk}{2\pi} \cos(\underbrace{k(x - c_0 t)}_{\text{"}O(1)\text{"}} + \gamma t k^3)$$

convergence is guaranteed because of k^3 (it is not convergent if we have only k)

NO FURTHER SIMPLIFICATION OF THIS INTEGRAL POSSIBLE



$x - c_0 t = $ "$O(1)$" we want this to be order 1 because we want the whole range of solutions (oscillatory + decaying)

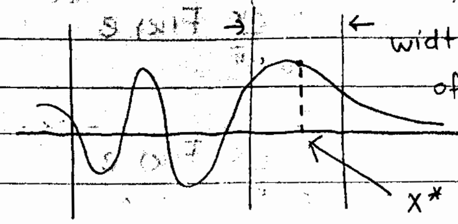
"Airy Integral"

$$Ai(z) = \frac{1}{\pi} \int_0^{\infty} \cos\left(kz + \frac{1}{3}k^3\right) dk$$

↓ relation with Bessel functions

$K_{1/3}\left(\frac{2}{3} z^{3/2}\right)$: modified "Hankel function" - Bessel-like function

$$I(x,t) = F(0)\,\frac{1}{L_{k^*}\,(3\gamma t)^{1/3}}\,A_i\left(\frac{x - \omega t}{(3\gamma t)^{1/3}}\right)$$



Assymptotic behavior

$$A_i(z) \sim \begin{cases} \dfrac{1}{2\sqrt{\pi}}\, z^{-\frac{1}{4}} \exp\left(-\dfrac{2}{3} z^{3/2}\right), & z \to +\infty \quad (z>0) \\ \dfrac{1}{\sqrt{\pi}}\, |z|\ \sin\left(\dfrac{2}{3}|z|^{3/2} + \pi/4\right), & z \to -\infty \end{cases}$$

W'' ~ max velocity by which velocity can propagate in the medium

W' ~ group velocity

W''' ~ damping / cutoff frequency

plasma: charges have their own frequency
(if the external forcing is too high frequency the charges may not be able to follow, the wave decays)

Back to the Stationary Phase Formula:

$$I(x,t) \approx \sum_{\bar{k}_i} F(\bar{k}_i)\,\frac{1}{\sqrt{2\pi\,|t\,W''(\bar{k}_i)|}}\, e^{-i\phi(\bar{k}_i) - i\frac{\pi}{4}\,\mathrm{sgn}(t\,W''(\bar{k}_i))}$$

↳ if we have more than 1 point $\bar{k}$

Any wave of this form can be written as

$$I(x,t) = \underbrace{A(x,t)}_{\text{amplitude}}\, e^{i\theta(x,t)}$$

amplitude ↳ $= \dfrac{F(\bar{k})}{\sqrt{2\pi\,|t\,W''(\bar{k})|}}\ \underbrace{e^{-i\,\pi/4\,\mathrm{sgn}(t\,W''(\bar{k}))}}_{\text{just a constant}}$

$\theta(x,t) = -\phi(\bar{k}) = \bar{k}x - W(\bar{k})t$ "phase"

Show that $I(x,t)$ is a slowly-varying in space and time
i.e. amplitude and phase are slowly varying
i.e. $\bar{k}$ is slowly varying

$\theta_x = \bar{k}_x x + \bar{k} - W'(\bar{k})t\,\bar{k}_x$ local wave number

$= \bar{k} + \underbrace{[x - W'(\bar{k})t]}_{0}\bar{k}_x \Rightarrow \bar{k} = \theta_x$

space derivative of phase

$\theta_t = \bar{k}_t x - W(\bar{k}) - W'(\bar{k})\bar{k}_t t$ local frequency

$= -W(\bar{k}) + \bar{k}_t\underbrace{(x - W'(\bar{k})t)}_{0} \Rightarrow W(\bar{k}) = -\theta_t$

time derivative of phase

We want to find $\bar{k}_x = ?$, $\bar{k}_t = ?$

$$W'(\bar{k}) = \frac{x}{t} \xRightarrow{\frac{\partial}{\partial x}} \bar{k}_x = \frac{1}{t\,W''(\bar{k})} = \frac{W'(\bar{k})}{W''(\bar{k})}\frac{1}{x}$$

$$\left|\frac{\bar{k}_x}{\bar{k}}\right| = \left|\frac{W'(\bar{k})}{\bar{k}\,W''(\bar{k})}\frac{1}{x}\right| : \text{"small"} \quad W'' \neq 0 \quad (x \to \infty)$$

$\Rightarrow$ $\bar{k}$ varies "slowly" with x

$$\xRightarrow{\frac{\partial}{\partial t}} \left|\frac{\bar{k}_t}{\bar{k}}\right| = \left|-\frac{W'(\bar{k})}{\bar{k}\,W''(\bar{k})}\frac{1}{t}\right| : \text{"small"}$$

$\Rightarrow$ $\bar{k}$ varies "slowly" in time

"Sneak preview" For other PDE's

we can assume solutions $u \approx A(x,t)\,e^{i\theta(x,t)}$

slowly-varying in x,t