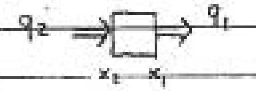
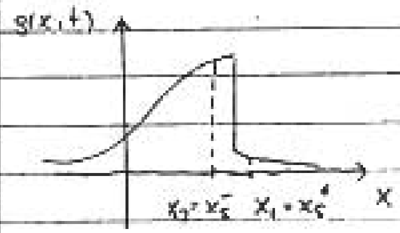
Where is the "jump"? $x_s = x_s(t)$

Recall: conservation law $\frac{d}{dt}\int_{x_2}^{x_1} g(x,t)\,dx = q(x_2,t) - q(x_1,t)$
(integration form)



in order to obtain the PDE, we assumed that $g \in C^1$
(differentiable)

Solutions to the integration form are called WEAK solutions of PDE.
The shockwave is a weak solution - it doesn't satisfy the PDE at all points.



- $\int_{x_2}^{x_1} \cdots = \int_{x_2}^{x_s(t)} \cdots + \int_{x_s(t)}^{x_1} \cdots$

- Recall: $\frac{d}{dt}\int_{a(t)}^{b(t)} dx\, g(x,t) = b'(t)\, g(b,t) - a'(t)\, g(a,t) + \int_{a(t)}^{b(t)} dx\, \frac{\partial g}{\partial t}(x,t)$

Conservation law:

$$x_s'(t)\, g(x_s^-,t) - x_s'(t)\, g(x_s^+,t) + \int_{x_1}^{x_2} dx\, \frac{\partial g}{\partial t} = q(x_2,t) - q(x_1,t)$$

take $x_2 \to x_1$

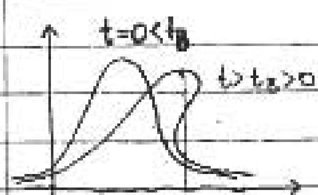
$$x_s'(t) = \frac{q(x_2,t) - q(x_1,t)}{g(x_2,t) - g(x_1,t)} = \frac{q(x_s^-,t) - q(x_s^+,t)}{g(x_s^-,t) - g(x_s^+,t)} \Rightarrow U = \frac{q_2 - q_1}{g_2 - g_1}$$

values right and left of the shock

February 23, 2004 Lecture 6

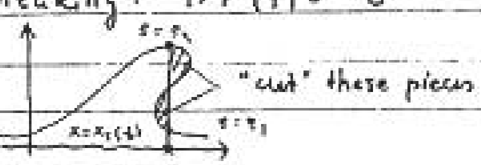
Reading Whitham 5.1-5.4
Review session on Transforms: FRI 4-5pm

Review IVP $\begin{cases} g_t + \underbrace{c(g)}_{Q'(g)} g_x = 0 \\ g(x,0) = f(x);\ x = \xi \end{cases}$ $F(\xi) = c(f(\xi))$
→ assume $F'(\xi) < 0$ somewhere



wave breaks for $t > t_B$, $t_B = 1/|F'(\xi)|_{max}$

Condition for breaking: $1 + F'(\xi)\, t = 0$



the pos cannot be arbitrary

Remedies: (A) Place "shock":
Restriction: "shock speed" $\frac{dx_s}{dt} = \frac{Q(g_1) - Q(g_2)}{g_1 - g_2}$

- the shock moves at speed prescribed by the conservation
- what we remove = what we add (conservation of mass)

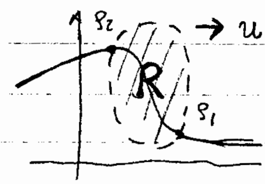
Ⓑ Modify the PDE

$q = Q(\rho) \to q = Q(\rho, \rho_x) = Q(\rho) - \nu \rho_x$, $\nu > 0$

New PDE: $(\rho_t + q_x = 0)$

$$\rho_t + \underbrace{\underbrace{Q'(\rho)}_{c(\rho)}\,\rho_x}_{\substack{\text{convection}\\ \text{term}\\ \text{(non-linear)}}} = \underbrace{\nu \rho_{xx}}_{\text{diffusion}}$$

the PDE is the interplay between diffusion and convection



smooth sol. but varies rapidly

☆ Assume a particular solution:

$\rho = \rho(x - Ut)$, traveling wave : if you move with speed U $(x - Ut = \text{const})$, you see $\rho = \text{const}$

↳ $U = \text{const}$

Want to find U s.t. ρ satisfies new PDE with $\rho = \rho_1, \rho_2$
(we see $\rho = \text{const} = \rho_1$ on left, $\rho = \rho_2 = \text{const}$ on right)

call $X = x - Ut$

$\rho_t = \frac{\partial}{\partial t}\rho(X) = -U \cdot \rho'(X)$, $\rho_x = \rho'(X)$, $\rho_{xx} = \rho''(X)$

$\Rightarrow -U\rho'(X) + Q'(\rho)\rho'(X) = \nu \rho''(X)$ (an ODE)

$$\frac{d}{dX}\left[Q(\rho) - U\rho\right] = \nu \frac{d^2\rho}{dX^2}$$

$$Q(\rho) - U\rho + \underset{\text{const}}{A} = \nu \frac{d\rho}{dX}$$

condition for ρ from the picture: $\begin{cases} X \to +\infty & \rho \to \rho_1 \\ X \to -\infty & \rho \to \rho_2 \end{cases}$

$X \to +\infty$ $Q(\rho_1) - U(\rho_1) + A = 0$

$X \to -\infty$ $Q(\rho_2) - U(\rho_2) + A = 0$ $\Rightarrow$ $U = \dfrac{Q(\rho_1) - Q(\rho_2)}{\rho_1 - \rho_2}$

How narrow is this region R?

→ Must specify $Q(\rho) = \alpha\rho^2 + \beta\rho + \gamma$, $\alpha > 0$ (we can assume this without lost of generality)

Take a traveling wave solution: observer that moves with the speed of the shock and sees a constant density.

Take $\rho = \rho(x - \mathcal{U}t)$ $\quad \underbrace{Q(\rho) - \mathcal{U}\rho}_{\text{quadratic}} + A = \nu \rho'(x)$

$$\begin{cases} Q(\rho_1) - \mathcal{U}\rho_1 + A = 0 \\ Q(\rho_2) - \mathcal{U}\rho_2 + A = 0 \end{cases} \Rightarrow Q(\rho) - \mathcal{U}\rho + A = \alpha(\rho-\rho_1)(\rho-\rho_2)$$

↳ these are the roots

$$\alpha(\rho-\rho_1)(\rho-\rho_2) = \nu\rho'(X) \Rightarrow \frac{d\rho}{\alpha(\rho-\rho_1)(\rho-\rho_2)} = \frac{dX}{\nu} \Rightarrow -\int \frac{d\rho}{\alpha(\rho_2-\rho)(\rho-\rho_1)} = \int \frac{dX}{\nu}$$

"+" "+"

$$\boxed{\frac{\rho_2-\rho}{\rho_1-\rho} = C e^{\alpha\frac{(\rho_2-\rho_1)X}{\nu}}} \quad \rho_2 > \rho_1$$

$\rho \to \rho_1 \Rightarrow$ LHS $\to \infty \Rightarrow$ RHS $\to \infty$ $\quad \Rightarrow X \gg \dfrac{\nu}{\alpha(\rho_2-\rho_1)}$

$\rho \to \rho_2 \Rightarrow$ LHS $\to 0 \Rightarrow$ RHS $\to 0$

$\Rightarrow X < 0$ and $|X| \gg \dfrac{\nu}{\alpha(\rho_2-\rho_1)}$

Estimate of width of $\mathcal{R}$: $O\left(\dfrac{2\nu}{\alpha(\rho_2-\rho_1)}\right)$

when $\nu \to 0$ (diffusivity) the width of the region goes linearly to 0 (shock) → BL

$\rho_t + q_x = 0$

☆ $Q(\rho)$: quadratic $\quad \rho_t + c(\rho)\rho_x = \nu\rho_{xx}$ $\qquad c(\rho) = Q'(\rho)$

multyply by $c'(\rho)$ $\qquad q = Q(\rho) - \nu\rho_x$

it involves lower derivative of ρ

$$\underbrace{c'(\rho)\rho_t}_{\frac{\partial}{\partial t}c(\rho)} + \underbrace{c'(\rho)c(\rho)\rho_x}_{c\frac{\partial}{\partial x}c(\rho)} = \underbrace{c'(\rho)\nu\rho_{xx}}_{\frac{d^2}{dx^2}c(\rho)}$$

$$\frac{\partial^2}{\partial x^2}c(\rho) = \frac{\partial}{\partial x}\left(c'(\rho)\rho_x\right) = c''\rho_x^2 + c'\rho_{xx}$$

for Q quadratic this is zero

$$\Rightarrow \boxed{\frac{\partial}{\partial t}c(\rho) + c(\rho)\frac{\partial}{\partial x}c(\rho) = \nu\frac{\partial^2 c}{\partial x^2}}$$

Burger's equation
- non-linear PDE → can be converted to linear PDE
- simpler PDE that involves diffusion and nonlinearity

System of PDE's

$\boxed{\text{Ex.}}$ $u_{tt} - \gamma u_{xx} = 0$ (wave equ. $\gamma = c^2 > 0$)

Let's convert it to systems of PDE's:

$$\begin{cases} v \equiv u_x \\ w \equiv u_t \end{cases} \text{ definition } \Rightarrow \begin{cases} v_t = w_x \\ w_t - \gamma v_x = 0 \end{cases}$$ (assuming some continuity

"second order PDE" $\rightarrow$ "system of first order PDE"
maybe we can apply the methods we learned about 1st order PDE

Recall: $a u_x + b u_y = 0$ (quasi-linear)

CHAR: $\begin{cases} \delta x = a\varepsilon \\ \delta y = b\varepsilon \end{cases} \Rightarrow$ along CHAR, $\delta u = 0$
(the direction of the CHAR are known)

For PDE system is not known a priori. The direction of the CHAR <u>must</u> be determined.

Take $\begin{cases} \delta x = \alpha\varepsilon \\ \delta y = \beta\varepsilon \end{cases}$ (α, β) to be determined

$$\delta v = (v_x \delta x + v_t \delta t) = (v_x \alpha + v_t \beta)\varepsilon$$
$$\delta w = \quad - " - \quad = (w_x \alpha + w_t \beta)\varepsilon$$

Good for quasi-linear.

We require that along CHAR, $m_1 \delta v + m_2 \delta w = 0$ (take this as def.)
(impose $\delta v = 0$ and $\delta w = 0$ is too strong, we will loose solutions ..)

superposition of variations zero along CHAR $\rightarrow$ (I) $m_1(v_x\alpha + v_t\beta) + m_2(w_x\alpha + w_t\beta) = 0$

superposition of equations along CHAR $\rightarrow$ (II) $\ell_1(v_t - w_x) + \ell_2(w_t - \gamma v_x) = 0$

Compare coefficients in front of same derivatives

$v_t: \quad \ell_1 = m_1\beta$
$w_x: \quad -\ell_1 = m_2\alpha$
$v_x: \quad -\ell_2\gamma = m_1\alpha$
$w_t: \quad \ell_2 = m_2\beta$

$$\Rightarrow \begin{cases} \ell_1\alpha = -\ell_2\gamma\beta \\ -\ell_1\beta = \ell_2\alpha \end{cases}$$

$$\begin{pmatrix} -\beta & -\alpha \\ \alpha & \gamma\beta \end{pmatrix}\begin{pmatrix} \ell_1 \\ \ell_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} \alpha & \gamma\beta \\ \beta & \alpha \end{pmatrix}\begin{pmatrix} -\ell_1 \\ \ell_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

non-trivial solution $\Rightarrow$ $\alpha^2 - \beta^2\gamma = 0$ $\Rightarrow$ $\alpha = \pm\beta\sqrt{\gamma}$

Without loss of generality let's take $\beta = 1$ $\Rightarrow$ $\alpha = \pm\sqrt{\gamma}$
we find that the direction of the CHAR
on which superposition of the solutions = const is given by $\pm\sqrt{\text{speed}}$

February 25, 2004 Lecture 7

Pick up:
- Handout 5
- Hmwk 2
- Solution to Hmwk 1
- Practice set 2

Review session on Friday 4-5pm.

PDE systems: THEME. PDE $\Rightarrow$ ODE(s)

Ex. $u_{tt} - \gamma u_{xx} = 0 \underset{\substack{v = u_x \\ w = u_t}}{\Longleftrightarrow} \begin{cases} v_t - w_x = 0 \\ w_t - \gamma v_x = 0 \end{cases}$

Matrix form: $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} v_t \\ w_t \end{pmatrix} + \begin{pmatrix} 0 & -1 \\ -\gamma & 0 \end{pmatrix}\begin{pmatrix} v_x \\ w_x \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

Steps:

① Define CHAR: $\begin{cases} \delta x = \alpha\varepsilon \\ \delta t = \beta\varepsilon \end{cases}$

② Combine PDE sys. into 1 eqn. $\ell_1(v_t - w_x) + \ell_2(w_t - \gamma v_x) = 0$ any ℓ_1, ℓ_2
Read last equation as a statement about variations $\delta v, \delta w$ along CHAR

Eq. (II) : $m_1\delta v + m_2\delta w = 0 = m_1(v_x\alpha + v_t\beta) + m_2(w_x\alpha + w_t\beta)$

Eq. (I) : $\ell_1(v_t - w_x) + \ell_2(w_t - \gamma v_x) = 0$

Compare coefs. of v_t, v_x, w_t, w_x to find m_1, m_2, ℓ_1, ℓ_2

$$\Rightarrow \begin{cases} \ell_1 = m_1\beta \\ -\ell_1 = m_2\alpha \\ \ell_2 = m_2\beta \\ -\gamma\ell_2 = m_1\alpha \end{cases} \qquad (\ell_1 \ \ell_2)\begin{pmatrix} \beta & \alpha \\ \alpha & \beta\gamma \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{matrix} \beta\ell_1 + \ell_2\alpha = 0 \\ \alpha\ell_1 + \beta\gamma\ell_2 = 0 \end{matrix}$$

$$\Rightarrow \begin{vmatrix} \beta & \alpha \\ \alpha & \beta\gamma \end{vmatrix} = 0 \iff \alpha = \pm\sqrt{\gamma} \quad \text{if } \beta = 1$$