May 3, 2004 Lecture 25

Optional reading: Hinch 1.1-1.4 4.1-4.3 5.1-5.4

Review of similarity

Stretching transformations (ST):

$(x,t,u) \to (\hat{x},\hat{t},\hat{u})$:

$\hat{x} = \lambda^{\alpha} x$, $\hat{t} = \lambda^{\beta} t$, $\hat{u} = \lambda^{\gamma} u$

$\lambda > \text{arb.} > 0$, $\alpha, \beta, \gamma$: real

PDE is invariant under ST if PDE-form in $(x,t,u)$ same as PDE-form in $(\hat{x},\hat{t},\hat{u})$

Idea:

(i) Find relations among $\alpha, \beta, \gamma$ that leave PDE invariant.

(ii) Construct desired solutions for these $\alpha, \beta, \gamma$ that are themselves invariant

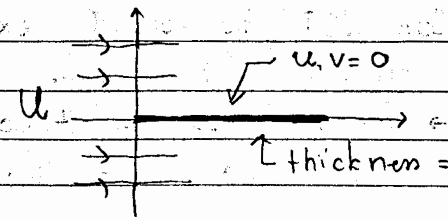
Similarity solutions: Particular solutions to PDE that respects invariance under ST

Particular solutions: not valid everywhere at every time.

Remark: Original conditions to PDE not always invariant under ST. Choose conditions that are also invariant.

**Example: Incompressible Fluid flow past an infinite plate**

$$\begin{cases} u: \text{ fluid velocity in } x \\ v: \text{ fluid velocity in } y \\ \nu: \text{ viscosity (const)} \end{cases}$$




PDE sys: $\begin{cases} u u_x + v u_y = \nu u_{yy} \\ u_x + v_y = 0 \end{cases}$ momentum conservation; mass conservation

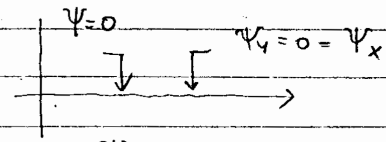
Conditions:

$$\begin{cases} u \to U & y \to \infty, \ x > 0 \\ u = U & x = 0, \ y > 0 \end{cases}$$

Reduce to a single equation by introducing a streamline function $\psi$:

$$\begin{cases} u = \psi_y \\ v = -\psi_x \end{cases} \overset{① \text{ PDE}}{\Longrightarrow} \text{PDE for } \psi \text{ is } \psi_y \psi_{xy} - \psi_x \psi_{yy} = \nu \psi_{yyy}$$

② satisfied



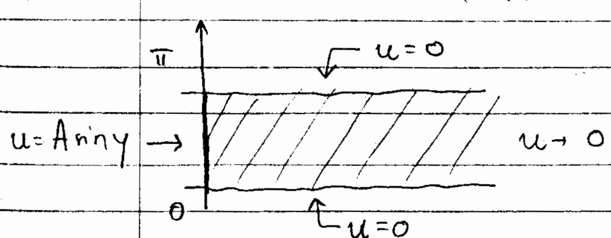

$\psi_y \to U, \ y \to \infty$

try similarity solutions
ODE is the Blasius equation

Perturbation theory for PDE's:

To help obtain approximate solutions when PDE's (or condns) contain small (non-dimensional) parameters

Ex. $\nabla^2 u + \varepsilon \left(\frac{\partial u}{\partial y}\right)^2 = 0, \quad u = u(x,y)$




Only method that can work on this problem is Green's function, but we want to use perturbation theory.

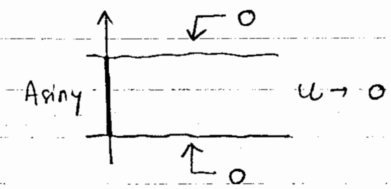
The PDE is not linear.
Can NOT be solved exactly.

→ For perturbation theory, everything has to be non-dimensional. $\epsilon$ has to be non-dimensional.

Step 1 : Set $\epsilon = 0$ and apply all given conditions. If solution exists in this case, then we have regular perturbation theory.

REGULAR PERTURBATION

$$\nabla^2 u = 0$$




separation of variables : obvious choice
(FT wants $x \in (-\infty; +\infty)$)

Try a solution of the form:
Comment: additive separation of variables will not work
→ problem get overdetermined

multiplicative : $\begin{cases} u = X(x) \cdot Y(y) \\ + \text{homog. bc's} \end{cases}$

$$\frac{X''}{X} = -\frac{Y''}{Y} = \text{const} > 0 \quad \text{(oscillatory behavior in } y\text{)}$$
$$= k^2 > 0 \quad (k>0)$$

$$X(x) = Be^{kx} + Ce^{-kx}$$
$$Y(y) = D\sin(ky) + E\cos(ky)$$

Homog. condns:
$u = 0$ at $y = 0$ ⇒ $E = 0$
$u = 0$ at $y = \pi$ ⇒ $D\sin(\pi k) = 0$ $k$ integer $k = n = 1, 2, ..$
$u \to 0$ at $x \to +\infty$ ⇒ $B = 0$

$u = X(x)Y(y) = \tilde{B} \sin(ny)\, e^{-nx} \quad (k=n) \quad \tilde{B} = DC$

Linear supperposition: $u^{(0)} = \sum_{n=0}^{+\infty} \tilde{B}_n e^{-nx} \sin(ny)$

$u^{(0)}$ : $u \approx u^{(0)} = u(\vec{r}\,;\, \varepsilon = 0)$

Apply condition at $x=0$ : $u\big|_{x=0} = A \sin y = \sum_{n=0}^{+\infty} \tilde{B}_n \sin(ny)$

$\Rightarrow \tilde{B}_n = \begin{cases} A, & n=1 \\ 0, & \text{else} \end{cases} \qquad \tilde{B}_1 = A$

Final solution: $u \simeq u^{(0)} = A e^{-x} \sin y$

Step 2 Consider $\varepsilon \neq 0$ to leading order

Assume $u(x,y;\varepsilon) = u^{(0)} + \varepsilon u^{(1)} + \varepsilon^2 u^{(2)} + \dots$

(for a regular perturbation, dependence on $\varepsilon$ is in power series)

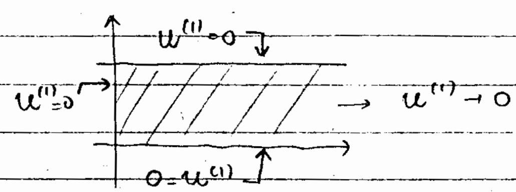
$\rightarrow$ $u$ has a "regular" expansion around $\varepsilon = 0$.

Questions
- ? is it really a convergent expansion?
- ? can we use it even if it is not convergent?

PDE for $u^{(1)}$ : $\nabla^2 u = -\varepsilon \left(\frac{\partial u}{\partial y}\right)^2 \Rightarrow \nabla^2 u^{(1)} = - \left(\frac{\partial u_0}{\partial y}\right)^2$

$\nabla^2 u^{(1)} = - \underbrace{\left(\frac{\partial u^{(0)}}{\partial y}\right)^2}_{\text{known}}$ (set equal coefficients of same powers of $\varepsilon$)

$\nabla^2 u^{(1)} = - A^2 e^{-2y} \cos^2 y$




zero conditions everywhere but the PDE is not homogeneous

(Poisson equation)

All higher order terms satisfy the Poisson equation.

Thus, we get :

a non-linear PDE with $\varepsilon \underset{\text{(reduces to)}}{\Rightarrow}$ sequence of linear PDE's.

Digression: To solve a non-homogeneous, linear PDE with homogeneous bc's we expand the solution to functions that satisfy homogeneous corresponding PDE with homogeneous bc's

suitable hom. PDE : Laplace
hom. bc's in $y$ !

Expand $u^{(1)}$ in Fourier series: (from step 1)

$$u^{(1)} = \sum_{n=1}^{\infty} a_n(x) \sin(ny)$$

PDE: $$\sum_{n=1}^{\infty} \left(a_n''(x) - n^2 a_n\right) \sin(ny) = -A^2 e^{-2x} \cos^2 y$$

RHS: $$-A^2 e^{-2x} \underbrace{\boxed{\sum_{n=1}^{+\infty} b_n \sin(ny)}}_{= \cos^2 y}$$

$$\Rightarrow b_n = \frac{2}{\pi} \int_0^{\pi} dy \, \underbrace{(\cos^2(y))}_{\frac{1+\cos 2y}{2}} \sin(ny) = \begin{cases} 0, & n: \text{even} \\ \frac{4}{\pi} \frac{n^2-2}{n(n^2-4)}, & n: \text{odd} \end{cases}$$

$$\Rightarrow \sum_{n=1}^{+\infty} \left[a_n''(x) - n^2 a_n(x)\right] \sin(ny) = -A^2 e^{-2x} \sum_{n=1}^{+\infty} b_n \sin(ny)$$

ODE for $a_n(x)$:

$\Rightarrow$ $n$: even $\quad a_n''(x) - n^2 a_n(x) = -A^2 e^{-2x} \, 0 = 0$

$n$: odd $\quad a_n''(x) - n^2 a_n(x) = -A^2 e^{-2x} \frac{4}{\pi} \frac{n^2-2}{n(n^2-4)}$

$$x \in (0, +\infty) \quad \begin{cases} a_n(x=0) = 0 \\ a_n(x \to \infty) = 0 \end{cases}$$

$$\boxed{a_n''(x) - n^2 a_n(x) = -A^2 e^{-2x} b_n}$$

$$a_n(x) = F e^{-nx} + G e^{nx} + M e^{-2x}$$

particular
will fail for $n=2$ !!!
but does not matter here

- $G = 0$
- $F + M = 0$

M is found by substitution in PDE: $(4-n^2)M = -A^2 b_n \Rightarrow \boxed{M = \frac{A^2 b_n}{n^2-4}}$

$$\Rightarrow \boxed{F = -\frac{A^2 b_n}{n^2-4}}$$

$$\Rightarrow \quad a_n(x) = \frac{A^2 b_n}{n^2-4}\left(e^{-2x} - e^{-nx}\right) \qquad b_n = \begin{cases} 0 & n: \text{even} \\ \frac{4}{\pi}\frac{n^2-2}{(n^2-4)n} & n: \text{odd} \end{cases}$$

$$u^{(1)}(x,y) = \frac{4A^2}{\pi} \sum_{n:\text{odd}} \frac{n^2-2}{(n^2-4)^2}\frac{1}{n}\left(e^{-2x} - e^{-nx}\right)\sin(ny)$$

$$u(x,y) = u^{(0)}(x,y) + \varepsilon u^{(1)}(x,y) + \dots \qquad |\varepsilon| \ll 1$$

Issues about convergence:

- Does the series $u = u^{(0)} + \varepsilon u^{(1)} + \dots + \varepsilon^n u^{(n)} + \dots$ converge?

Answer: Yes, by theory of integral equations.
↳ for sufficiently small $\varepsilon$, $|\varepsilon| < |\varepsilon_0|$

- More generally, do series of such form converge?

Answer: No, in principle; They may be divergent, yet assymptotic.

(assymptotic series: you fix $n$ and you look if your finite sum is close to the true $u$ if $\varepsilon \ll 1$)

May 5, 2004 Lecture 26

- Quiz 2: Mon. May 10, pick up in class
  to be returned Wed. May 12 before class
- Review Session: PRI 4-5:30 pm

Review regular perturbations for PDE
(advice: have eqns. in non-dimensional form)

General problem: PDE: $\begin{cases} F(x, x_2, \dots x_n, \text{deriv. of } u; \varepsilon) = 0, & |\varepsilon| \ll 1 \\ + \text{condns} \end{cases}$