18.306: Advanced Partial Differential Equations with Applications
M.I.T. Department of Mathematics **Spring 2004**
Practice Set 4 **Handed out**: Monday, 04/05/04
Due: NEVER

You are NOT required to return solutions. Some of these problems may be given in Quiz 2. The description of the references given in parentheses can be found in the Bibliography for 18.306.

31. (Carrier & Pearson, § 1.7, pp. 17, 18.) This problem gives a systematic way to treat a nonhomogeneous PDE with nonhomogeneous boundary (in x) and initial (in t) conditions. Describe how you would solve the diffusion equation with a source term, $u_t = \nu u_{xx} + w(x,t)$, where $u = u(x,t)$, with the conditions $u(x,0) = f(x), u(0,t) = g(t), u(l,t) = h(t)$, where $0 < x < l$ and $t > 0$. **Hint:** Try to convert the given problem to one with nonhomogeneous PDE satisfying homogeneous boundary conditions. For instance, you may introduce ψ by $\psi = u - \{g(t) + (x/l)[h(t) - g(t)]\}$. Show that ψ satisfies homogeneous boundary conditions and derive a new PDE for ψ. Try a Fourier series $\psi(x,t) = \sum_n a_n(t)\beta_n(x)$. What is $\beta_n(x)$ and how would you find an ODE for $a_n(t)$? What is the condition for $a_n(t)$?

32. For given $\omega > 0$, derive an equation that governs the eigenvalues λ of the problem

$$\nabla^2 u + \omega^2 u = \lambda u; \qquad u(r,0) = u(a,\phi) = 0,\ u_\phi(r,\pi) = 0; \qquad r,\ \phi : \text{polar coordinates}$$

where $0 \le r < a$ and $0 < \phi < \pi$. What are the corresponding eigenfunctions?

33. This problem is an application of eigenvalue theory to statistical mechanics (Bose-Einstein condensation). Derive an equation for the energy levels per atom in a gas of pairwise interacting atoms enclosed in an one-dimensional (1D) box. The motion of each atom is governed by the nonlinear Schrödinger equation, $i\Psi_t(x,t) = -\Psi_{xx}(x,t) + a|\Psi(x,t)|^2\Psi(x,t) - \zeta\Psi(x,t),\ 0 < x < L$, with the boundary conditions $\Psi(0,t) = \Psi(L,t) = 0$ $(a,\ \zeta > 0)$. **Hint:** Set $\Psi = e^{-iEt}\psi(x)$ and describe E so that the given PDE and conditions are satisfied.

34. (Carrier & Pearson, § 3.11, pp. 52, 53.) This problem aims to familiarize you with a uniqueness proof. Give a proof that the one-dimensional (1D) wave equation with a space-dependent velocity c and a forcing term f, $u_{tt} - c(x)^2 u_{xx} = f(x,t)$, has a unique solution $u(x,t)$ in $0 < x < l, t > 0$ if u satisfies

$$u(0,t) = a(t), \quad u(l,t) = b(t), \quad u(x,0) = h(x), \quad u_t(x,0) = p(x).$$

Hint: Define the integral $I(t) = \frac{1}{2}\int_0^l dx\, [\phi_t^2/c^2 + \phi_x^2]$, where ϕ is any solution of the given PDE with $f \equiv 0$ that satisfies $\phi(0,t) = \phi(l,t) = 0$. Show that $dI/dt = 0$. Take $\phi = u - v$ where u and v satisfy the given PDE and all given conditions. Infer that $\phi \equiv 0$. *What is the physical meaning of I in the case of a stretched string?*

35. Find explicitly the Green function G for outgoing waves in 3D that is specified by the solution of the form $w = G(\mathbf{r};\mathbf{r}')e^{i\omega t}$ of the PDE $w_{tt} - c^2\nabla^2 w = -\delta(\mathbf{r} - \mathbf{r}')\, e^{i\omega t}$, where $\mathbf{r} = (x, y, z)$ is the position vector in the 3D infinite space.

36. Define and find the Green function $G(x,t;x',t')$ for the time-dependent Schrödinger equation $i\Psi_t(x,t)+\Psi_{xx}(x,t)=V(x,t)\Psi(x,t)$, with the initial condition $\Psi(x,0)=a(x)$. Convert the given initial-value problem into an integral equation. Show the connection with the Green function defined in class for the diffusion equation.

37. Define the Green function $G(x,y;x'y')$ for the nonhomogeneous PDE $u_{xxxx}-u_{yy}=\rho(x,y)$ in the strip $(-\infty<x<\infty,\ 0<y<1)$, where u satisfies prescribed boundary conditions at $y=0$ and $y=1$, and is bounded as $|x|\to\infty$. Calculate $G(0,\frac{1}{2};0,\frac{1}{2})$ in terms of a (convergent) single series. **Hint:** $G(x,y;x',y')$ should satisfy corresponding *homogeneous* conditions (why?). Use Fourier Transform in x and the Residue Theorem.

38. (Carrier & Pearson, Prob. 9.3.9, p. 147.) A two-dimensional (2D) Green's function $G(x,y;x',y')$ for the Laplacian ∇^2 in a finite region is defined by

$$G_{xx}+G_{yy}=\delta(x-x')\,\delta(y-y');\qquad G=0\quad\text{on boundary.}$$

Show, by using separation of variables, that for the case of the square region $0<x<l, 0<y<l$, G is given by

$$G=\sum_{n=1}^{\infty}\frac{\sin(n\pi x'/l)\sin(n\pi x/l)}{n\pi\sinh(n\pi)}\,\{\cosh[n\pi(l-y-y')/l]-\cosh[n\pi(l-|y-y'|)/l]\}.$$

39. (Levine, Chap. 5, Prob. 4, pp. 61, 62.) Show that the first-order nonlinear PDE $u_y+uu_x=0$, where x and y are space variables, has a general solution of the implicit form $u=F(x-uy)$, where F is arbitrary. On the other hand, dimensional considerations suggest a similarity form $u=H(x/y)$. Find H explicitly through an ODE in $\eta=x/y$ and verify its agreement with the exact solution of the PDE when $F(\xi)=\xi$ and $y\gg 1$.

40. (Levine, Chap. 5, Prob. 8, pp. 64–66.) The PDE $C_t=k(C^nC_x)_x$, where $C=C(x,t), k>0$ and n: integer, is relevant to certain models of thermal conduction with a nonlinear heat flux, and of filtration which concerns gas flows through porous media. Consider a solution C to this PDE that, at $t=0$, vanishes everywhere except in some neighborhood of the origin $x=0$, and also obeys the relation $\int_{-\infty}^{\infty}C(x,t)\,dx=Q=\text{const.}$, which often expresses conservation of total mass.

(a) Verify that the variables $(kt/Q^2)^{\frac{1}{n+2}}C$ and $\xi=x(Q^nkt)^{-\frac{1}{n+2}}$ are dimensionless.

(b) Find particular solutions of the form $C=[Q^2(kt)^{-1}]^{\frac{1}{n+2}}f(\xi)$. What is the ODE and constraints (conditions) satisfied by f? Is f even or odd? Find unique particular solutions for the following cases: (i) $n=1$, and (ii) $n=-1$. Notice the difference in the nature of the two solutions. How do these solutions behave as $t\to\infty$? **Hint:** Apply $C\to 0$ when $|x|$: suff. large (C: continuous), and the integral constraint involving Q in order to determine the integration constants of the ODE.

(c) Show that, if $C(x,t)$ is a solution of the given PDE, then $(\beta\alpha^{-2})^{1/n}\,C(\alpha x,\beta t)$ is also a solution, where α and β are constants and $\beta>0$.