April 14, 2004

Lecture 19

Opt. reading Whitham 11.1-11.6
Debnath 7.2-7.65

Pick up: New (Hmwk 5), Graded (Hmwk 3)

Review| Example
$$\begin{cases} u_{tt} - c^2 u_{xx} = \boxed{-\mu u_{xx} u_x^2}^{\,g}, & -\infty < x < +\infty,\ t>0;\ \mu>0 \\ u(x,0) = a(x) & u_t|_{t=0} = b(x) \end{cases}$$

Starting point:

$$u = u_h + u_p$$

$g=0$ → PDE with $g = -\mu u_{xx} u_x^2$ + homog. condns

(I) $\begin{cases} u_{h,tt} - c^2 u_{h,xx} = 0 \\ u_h : \text{same condns as } u \end{cases}$

↓

D'Alembert solution

(II) $u_p = \int_0^\infty dt' \int_{-\infty}^{+\infty} dx'\, G(x,t;x',t')\, g(x',t')$

G? $g \Rightarrow \delta(x-x')\,\delta(t-t')$

(II) $\begin{cases} G_{tt} - c^2 G_{xx} = \delta(x-x')\,\delta(t-t') \\ G|_{t<0} = 0, \quad G_t|_{t<0} = 0 \quad \text{(homogeneous conditions)} \end{cases}$

↳ only 1 path can satisfy these cnds!

(III) Calculate G: FT in x, t
+ contour integration (choice of path from conditions)

$\Rightarrow G(x,t;x',t') = 0, \quad t<t'$
↳ causal

$$G = \begin{cases} \frac{1}{2c}, & c(t-t') > (x-x') \\ 0, & \text{otherwise} \end{cases} \qquad t > t'$$

(IV) Integral Eqn for u:

$$u = u_h - \frac{\mu}{2c} \int_0^t dt' \int_{x-c(t-t')}^{x+c(t-t')} dx'\, u_{xx}(x',t')\, u_x^2(x',t')$$

(V) Solve iteratively if $\boxed{\mu \ll 1}$

A generalization (apply with caution, not always applicable)

linear operator

"General" PDE problem : $\begin{cases} \mathcal{L}u = g(\vec{r};\ldots) \\ + \text{condns (on } \partial R) \end{cases}$ scalar $\quad \vec{r} \in R \subset \mathbb{R}^n$

(fits the two examples we solved in class) $\quad \vec{r} = (x_1, \ldots x_n)$ n-dim where $\vec{r}$ includes all independent variables (space / time)

$(\vec{r} \equiv \vec{x})$

"Theorem" If $\mathcal{L}$ is self-adjoint then

$$u = u_h + \int d^n\vec{x}\, G(\vec{x};\vec{x}')\, g(\vec{x}';\ldots$$

$g = 0$ $\qquad$ $g(\vec{x}) \Rightarrow \delta(\vec{x}-\vec{x}')$

$\begin{cases} \mathcal{L}u_h = 0 \\ u_h \text{ same condns as } u \end{cases}$ $\qquad$ $\begin{cases} \mathcal{L}G = \delta(\vec{x}-\vec{x}') \\ + \text{ homogeneous condns} \end{cases}$

-> the problem can be restated as an integral equation that contains all the information in one line

-> not all physical problems are described by self-adjoint operators. There is a generalization of this approach for ADJOINT operators.

Dispersive Waves

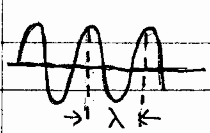
Hyperbolic equations (wave eqn. etc.) : have a particular feature
- discontinuities CAN propagate
- oscillations in physical space and time (both)

"Dispersive" systems : do NOT allow discontinuities to propagate
↳ refers to the nature of solutions (not the PDE)

<u>Ex. 1</u> $\quad u_{tt} - c^2 u_{xx} = 0 \quad$ (wave eqn. 1D)

Try solutions of fixed wavelength λ, $k = 2\pi/\lambda$

(sep. of var. where the x-part is a wave)



$u(x,t) = T(t) e^{ikx} \overset{PDE}{\Rightarrow} T'' + k^2 c^2 T = 0$ (ODE)

$\Rightarrow T(t) = e^{\pm ikct} = e^{-i\omega t}$ $\quad \omega = \pm kc$

$\downarrow$ frequency depends on k

What is the most general solution if $-\infty < x < \infty$? - superposition over all k

$$u(x,t) = \int_{-\infty}^{+\infty} \frac{dk}{2\pi} A(k) e^{-ikct + ikx} + \int_{-\infty}^{+\infty} \frac{dk}{2\pi} B(k) e^{ikct + ikx}$$

(Fourier integral, in fact a superposition)

Two conclusions:

1) Assuming a particular form of the solution we find a relation between ω and k

2) For this particular PDE (linear) we can make a superposition over all k.
The amplitudes $A(k)$ and $B(k)$ can be found from the initial data.

Same result if $\left(\frac{\partial}{\partial t} \Rightarrow -i\omega, \ \frac{\partial}{\partial x} \Rightarrow +ik\right)$

Ex. 2 $\left(\frac{\partial}{\partial t} + c_1 \frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} + c_2 \frac{\partial}{\partial x}\right) \cdots \left(\frac{\partial}{\partial t} + c_n \frac{\partial}{\partial x}\right) = 0$

How to find ω as a function of k?

$$(-i\omega + ic_1 k) \dots (-i\omega + ic_n k) = 0 \underset{\text{roots}}{\Rightarrow} \left\{ \begin{array}{l} \omega = c_1 k \\ \vdots \\ \omega = c_n k \end{array} \right.$$

previous example was a particular case of this

we have $\underbrace{\frac{\omega}{k}}_{\text{"phase velocity"}}$: constant

wave does not disperse

Ex.3 Klein-Gordon equation (membrane with restoring force / 0-spin particles)

$$u_{tt} - \alpha^2 \nabla^2 u + \beta^2 u = 0$$

find ω as fcn of κ: $\quad -\omega^2 + \alpha^2 \kappa^2 + \beta^2 = 0$

$$\omega = \pm\sqrt{\alpha^2\kappa^2 + \beta^2}\ , \quad \frac{\omega}{\kappa} \neq \text{const}$$

waves with diff κ propagate ~~at different speeds~~
-> waves disperse

Definition:

① The relation $\omega = W(\kappa)$ or $\omega = W(\vec{\kappa})$ is called dispersion relation.

② A "system" described by disp. relation $\omega = W(\kappa)$ is called strictly dispersive iff:

Ⓐ $W(\kappa)$ [or $W(\vec{\kappa})$] is real if κ (or κ_i components) are real.

Ⓑ $\frac{d^2W}{d\kappa^2} \neq 0$ almost everywhere

[$\det \left| \frac{\partial^2 W}{\partial \kappa_i \partial \kappa_j} \right| \neq 0$ almost everywhere]

Suppose we have linear PDE in 1D and this PDE give us $\omega = W(\kappa)$. Then we can apply superposition to get the solution of this PDE

Linear PDE for u in 1D $\Rightarrow \omega = W(\kappa) \Rightarrow$ superposition $(-\infty < x < +\infty)$

$$I(x,t) = \int_{-\infty}^{+\infty} \frac{d\kappa}{2\pi}\, A(\kappa)\, e^{i\kappa x - iW(\kappa)t}$$

↳ no oscillations

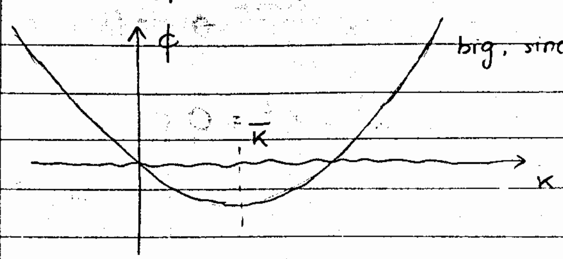
(the solution can be put in this form)

What's the form of I when $x \to \infty$, $t \to \infty$ and $\frac{x}{t}$ fixed?
(can be applied to non-linear PDE's but after carefully defining what is a disp. relation in this case)

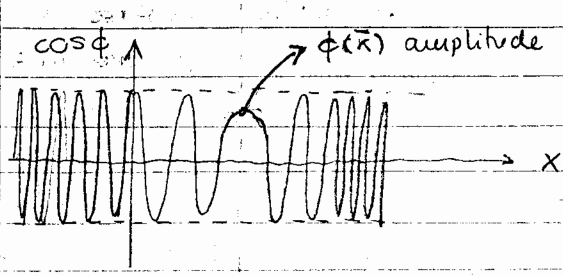
$$\text{phase} = i\kappa x - iW(\kappa)t = it\left[\kappa \frac{x}{t} - W(\kappa)\right] = -it\underbrace{\left[W(\kappa) - \kappa\frac{x}{t}\right]}_{\phi(\kappa; x,t) \,=\, t\,P(\kappa)}$$

Plot $\phi(\kappa; x,t)$ for x,t fixed (x,t large)

suppose $\phi(\kappa=0)=0$ and $\exists\, \bar{\kappa} : \phi'(\bar{\kappa})=0$



$$e^{-i\phi} = \cos\phi - i\sin\phi$$



The major contribution to κ-integration comes from a neighborhood of $\bar{\kappa}$.

$$\Rightarrow \text{Approximation} \begin{cases} A(\kappa) \approx A(\bar{\kappa}) \\ P(\kappa) \cong P(\bar{\kappa}) + \dfrac{(\kappa-\bar{\kappa})^2}{2} P''(\bar{\kappa}) \end{cases}$$

$$I(x,t) \simeq A(\bar{\kappa})\, e^{-itP(\bar{\kappa})} \underbrace{\int_{-\infty}^{+\infty} \frac{d\kappa}{2\pi} e^{-i\frac{t}{2}(\kappa-\bar{\kappa})^2 P''(\bar{\kappa})}}_{\underbrace{\int_{-\infty}^{+\infty} \frac{d\xi}{2\pi} e^{-i\alpha\xi^2}}_{\frac{e^{-i\frac{\pi}{4}\operatorname{sgn}(\alpha)}}{\sqrt{4\pi|\alpha|}}}} \qquad \alpha = \frac{t}{2} P''(\bar{\kappa}), \quad \xi = \kappa - \bar{\kappa}$$

$$\Rightarrow \boxed{I(x,t) \simeq \frac{A(\bar{\kappa})\, e^{-itP(\bar{\kappa}) - i\frac{\pi}{4}\operatorname{sgn}(P''(\bar{\kappa}))}}{\sqrt{2\pi t\, |P''(\bar{\kappa})|}}}$$

$\rightarrow$ the stationary phase method