18.306: Advanced Partial Differential Equations with Applications
M.I.T. Department of Mathematics **Spring 2004**
Homework 5 **Handed out**: Wednesday, 04/14/04
Due: Wednesday, 04/28/04

The description of the references in parentheses can be found in the Bibliography for 18.306.

15. The transmission of an electrical signal $u(x,t)$ in a weakly nonlinear medium is governed by the PDE $u_{tt} - u_{xx} = -\mu e^{-b|t|} u_{xx} u_x^2$, $-\infty < x, t < +\infty, \mu > 0$, $b > 0$. Consider a signal that behaves as a plane wave in the distant past, $u(x,t) \approx e^{-ikt+ikx}$ as $t \to -\infty$. (How many conditions for the PDE does this behavior amount to ?)

    (a) Find the Green function that can be used to convert the given nonlinear PDE into a (nonlinear) integral equation. Write down the integral equation explicitly.

    (b) For nonzero $\mu \ll 1$, calculate u approximately. Your answer should involve μ to the leading order.

16. (Drazin & Johnson, Prob. Q1.18, p. 19.) A particular type of electromagnetic waves in a dissipative medium are governed by the PDE $u_{tt} - c^2\, u_{xx} + 2\mu(u_t + a u_x) = 0$, where $c > 0, \mu > 0$, and a is real.

    (a) Find the relation $\omega = W(k)$ for a plane wave $u = e^{ikx - i\omega t}$ in this medium. Is this wave strictly dispersive? Explain. Calculate the phase and group velocities as functions of k.

    (b) Derive approximations to $W(k)$ for fixed k in the following cases: (i) Sufficiently small μ; retain terms to order μ and show that u decays if $c > a > -c$.
    (ii) Sufficiently large μ; retain leading terms, and show that no waves can grow in time. Show that this case is equivalent to the approximation of small k (i.e., long waves) for fixed μ.

17. Consider the Boussinesq PDE, $u_{tt} - \alpha^2 u_{xx} = \beta^2 u_{xxtt}$, which describes sufficiently long water waves.

    (a) Derive the dispersion relation $\omega = W(k)$ if α and β are positive constants. What is the group velocity v_g? What is the maximum value $|v_g|_{max}$ of $|v_g|$? What type of wavepackets ("information") travel faster in such a medium?

    (b) In (a), give a precise condition on t and x for the stationary-phase method to be valid. Accordingly, describe (mathematically) the waves for sufficiently large x and t.

    (c) Give the differential equations for the amplitude A and phase θ of a slowly-varying solution $u = Ae^{i\theta}$ when $\alpha = \mu \cdot |u|, \mu$: constant > 0, β: constant > 0, and hence the PDE becomes nonlinear. What is the meaning of a dispersion relation in this case?

18. (Levine, Chap. 5, Prob. 2, p. 59.) The steady-state irrotational flow of a fluid in 2D is described by the Laplace equation, $\Psi_{xx} + \Psi_{yy} = 0$; $\Psi = \Psi(x,y)$ is the scalar potential ($\mathbf{u} = -\nabla\Psi$ is the fluid velocity).

(a) Explain why one expects particular solutions of the form $\Psi = f(x/y)$, where x/y: similarity variable. Explain why such solutions should depend only on the polar angle ϕ (where r, ϕ: polar coordinates). Find f explicitly via expressing the Laplace equation in polar coordinates.

(b) Now try particular solutions of the form $\Psi = x^\gamma f(x/y)$, where γ is a constant. Find the ODE that $f(\xi)$ satisfies, where $\xi = x/y$. So far, γ can take any value. Solve the ODE for $\gamma = 1$.

Bonus question: Explain what happens if $\Psi = x^\gamma f(x/y)$ is forced to obey the conditions $\Psi \to 0$ as $y \to +\infty$ (x: fixed), and $\Psi_x - \Psi_y = 0$ on $y = x$. (The last condition imposes zero velocity normal to the line $y = x$.) Is γ still free to take any value? Calculate Ψ explicitly.