Non-dimensional quantities: $\frac{x}{(mkt)^{1/3}}$, $\rho\left(\frac{kt}{m^2}\right)^{1/3}$

Similarity solution:

$$\rho\left(\frac{kt}{m^2}\right)^{1/3} = h\left(\frac{x}{(mkt)^{1/3}}\right) = h(\xi)$$

h ↳ to be found. ξ ↳ $\frac{x}{(mkt)^{1/3}}$

April 30, 2004 Lecture 24

Pick up solution to Hmwk 5

Similarity solution: Ex, nonlinear diffusion $(\nu = k\rho^n)$ $(q = -\nu\rho_x)$

$$\begin{cases} \rho_t = k(\rho^n \rho_x)_x \\ \rho \to 0 \text{ "fast" as } |x| \to +\infty \\ \rho(x, t=0) = f(x) \end{cases} \Rightarrow \int_{-\infty}^{+\infty} dx\, \rho(x,t) = m = \text{const} = \int_{-\infty}^{+\infty} f(x)\,dx$$

$\boxed{\text{Method A}}$ $n=1$

Quantity	ρ	x	t	k	m
Dimension	$[\rho]$	L	T	$\frac{L^2}{T}\frac{1}{[\rho]}$	$[\rho]L$

Non-dimensional parameters $\rho\left(\frac{kt}{m}\right)^{1/3}$ $\xi \equiv \frac{x}{(mkt)^{1/3}}$

Similarity Ansatz: $\rho\left(\frac{kt}{m}\right)^{1/3} = g(\xi)$ → to be found

Why?

- before we wrote first a function of 2 parameters and then we took one of them to be small and factored it out as a power;
- here since the equation is non-linear, the unknown function enters in the nondimensional parameters

$$u/u_0 = f\left(\frac{a}{\sqrt{\nu t}}, \frac{x}{\sqrt{\nu t}}\right) \qquad \text{Linear diffusion } u_t = \nu u_{xx}$$

Moral: dimensional analysis is powerful tool if you are experienced with the problem and know what to expect

Use stretched coordinates, otherwise.

$\boxed{\text{Method B}}$

Stretching transformations

$$\begin{cases} x' = \lambda^{\alpha} x \\ t' = \lambda^{\beta} t \\ \rho' = \lambda^{\gamma} \rho \end{cases} \qquad \begin{array}{l} \lambda > 0 \\ \alpha, \beta, \gamma : \text{to be found} \end{array}$$

(i) PDE is invariant under stretching transformation
find relation between α, β, γ so that the PDE remains invariant

PDE: in (x', t', ρ') :
$$0 = \frac{\partial \rho'}{\partial t'} - \kappa \frac{\partial}{\partial x'} \left(\rho' \rho'_{x'} \right)_{x'}$$
$$= \lambda^{\gamma - \beta} \frac{\partial \rho}{\partial t} - \lambda^{2\gamma - 2\alpha} \kappa \frac{\partial}{\partial x} \left(\rho \frac{\partial \rho}{\partial x} \right)$$

if PDE true in (x, t, ρ) it is true also in (x', t', ρ')
$$\Rightarrow \lambda^{\gamma - \beta} = \lambda^{2\gamma - 2\alpha} \quad \Rightarrow \quad \boxed{\gamma = 2\alpha - \beta} \quad (1)$$

second relation from the integral

$$m = \int_{-\infty}^{+\infty} dx\, \rho = \int_{-\infty}^{+\infty} dx'\, \rho'$$
stretching transformation to preserve total mass
$$= \lambda^{\alpha + \gamma} \int_{-\infty}^{+\infty} dx\, \rho \quad \Rightarrow \quad \boxed{\alpha + \gamma = 0} \quad (2)$$

$$(1), (2) \Rightarrow \begin{cases} \gamma = -\alpha \\ \beta = 3\alpha \end{cases} \qquad \begin{cases} x' = \lambda^{\alpha} x \\ t' = \lambda^{3\alpha} t \\ \rho' = \lambda^{-\alpha} \rho \end{cases}$$

(ii) Form invariant quantities from (x', t', ρ')

$$\lambda^{\alpha} = \frac{x'}{x} = \left(\frac{t'}{t} \right)^{1/3} = \left(\frac{\rho'}{\rho} \right)^{-1}$$

$$\Rightarrow \quad \frac{x'}{t'^{1/3}} = \frac{x}{t^{1/3}} \quad \text{and} \quad \frac{\rho'}{(t')^{-1/3}} = \frac{\rho}{(t)^{-1/3}}$$

(A) (B)

(iii) Write a similarity solution
solution that remains invariant under stretching transformations of steps (i) (ii)
→ invariance of PDE and of total mass

diffusion equation respects mass conservation because it comes from conservation law and ρ vanishes at ∞.

$$\frac{\rho}{(t)^{-1/3}} = \underbrace{h}_{\text{to be found.}}\left(\frac{x}{t^{1/3}}\right) \quad : \text{definition}$$

It is not exact, when is it valide?

Combine with method A
make LHS and argument non-dimensional

$$\frac{\rho}{(t)^{-1/3}} = h\left(\frac{x}{t^{1/3}}\right) \Rightarrow \rho = t^{-1/3}\, h\left(\frac{x}{t^{1/3}}\right)$$

$$\updownarrow$$

$$\rho\left(\frac{\kappa t}{m^2}\right)^{1/3} = g(\xi) \;,\; \xi = \frac{x}{(m\kappa t)^{1/3}}$$

$$\downarrow$$

$$\boxed{\rho = \left(\frac{m^2}{\kappa t}\right)^{1/3} g(\xi)}$$

$\boxed{\text{ODE for } g(\xi)\,?}$

PDE: $\dfrac{\partial \rho}{\partial t} = \kappa \dfrac{\partial}{\partial x}\left(\rho \dfrac{\partial \rho}{\partial x}\right)$

LHS: $\dfrac{\partial \rho}{\partial t} = -\dfrac{1}{3t}\left(\dfrac{m^2}{\kappa t}\right)^{1/3} g(\xi) + \left(\dfrac{m^2}{\kappa t}\right)^{1/3} g'(\xi)\left(-\dfrac{1}{3t}\xi\right)$

RHS: $x = (m\kappa t)^{1/3}\,\xi \Rightarrow \dfrac{\partial}{\partial x} = \dfrac{1}{(m\kappa t)^{1/3}}\dfrac{\partial}{\partial \xi}$

$$\kappa \frac{1}{(m\kappa t)^{2/3}}\left(\frac{m^2}{\kappa t}\right)^{2/3} \frac{\partial}{\partial \xi}\left(g \frac{\partial}{\partial \xi} g\right)$$

PDE

$$\Rightarrow -\frac{1}{3t}\left(\frac{m^2}{\kappa t}\right)^{1/3}\left[g + \xi g'\right] = \kappa \left(\frac{m^2}{\kappa t}\,\frac{1}{m\kappa t}\right)^{2/3} \frac{\partial}{\partial \xi}\left(g \frac{\partial}{\partial \xi} g\right)$$

$$\Rightarrow -\frac{1}{3}\Big[\underbrace{g(\xi)+\xi g'(\xi)}_{\frac{d}{d\xi}(\xi g(\xi))}\Big] = \frac{d}{d\xi}\left(g\frac{dg}{d\xi}\right) \qquad \text{ODE for } g$$

Conditions $g(\xi) \to 0$ fast as $|\xi| \to +\infty$

$$-\frac{1}{3}\xi g(\xi) = g\frac{dg}{d\xi} + \underbrace{\text{const}}_{=0} \quad \text{when taking limit to } \infty$$

$$\Rightarrow gg' + \frac{1}{3}\xi g = 0 \qquad \text{non-linear equation}$$

$$g\left[g' + \frac{1}{3}\xi\right] = 0$$

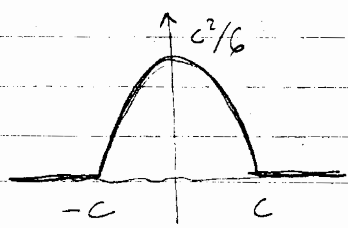
$$\Rightarrow g(\xi) = 0 \quad \text{or} \quad g' = -\frac{1}{3}\xi \Rightarrow g(\xi) = -\frac{\xi^2}{6} + \boxed{\text{const}} \quad \text{const} = c^2/6$$

$$= \frac{c^2-\xi^2}{6}$$

solution has two possible forms
we have to compromise these two conditions

g: continuous in ξ

$$g(\xi) = \begin{cases} \dfrac{c^2-\xi^2}{6}, & |\xi| < c \\ 0, & |\xi| > c \end{cases} \qquad \text{continuous and } g \to 0 \quad |\xi| \to +\infty$$



solution of compact support

linear diffusion equation: you start with a bump it gets instantaneously non-zero everywhere.

Find the constant c ?

integral condition:

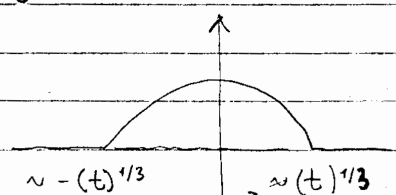
$$(mkt)^{1/3}\int_{-\infty}^{+\infty}\frac{dx}{(mkt)^{1/3}}\ \underbrace{\rho(x,t)}_{\left(\frac{m^2}{kt}\right)^{1/3} g(\xi)} = m \quad \Rightarrow \quad \boxed{\int_{-\infty}^{+\infty} d\xi\, g(\xi) = 1}$$

condition we have to apply to g

$$\int_{-c}^{c} d\xi\ \frac{c^2-\xi^2}{6} = 2\int_{0}^{c}\frac{c^2-\xi^2}{6}\,d\xi \Rightarrow 1 \quad \Rightarrow \quad c=\left(\frac{9}{2}\right)^{1/3}$$

$$\rho(x,t) = \left(\frac{m^2}{kt}\right)^{1/3}\cdot\begin{cases}\frac{1}{6}\left[\left(\frac{9}{2}\right)^{2/3} - \frac{x^2}{(mkt)^{2/3}}\right], & |x| < (9mkt/2)^{1/3}\\ 0, & |x| > (9mkt/2)^{1/3}\end{cases}$$

t: fixed



as time advances these points move away (their speed go to zero)

limit proportional to this

Solution is valid for "long, intermediate times"

- "length initial data" $\ll (mkt)^{1/3}$
- $x = O((mkt)^{1/3})$