March 9, 2004 Lecture 9

Pick up: Handout 6 (FT review) ; Graded Hmwk 1
Review session: FRI, 4:15-5pm
Quiz 1, MON March 15

Wave equation (1D) : $u_{tt} = c^2 u_{xx}$, $-\infty < x < \infty$, $t > 0$

General solution : $u(x,t) = F(x-ct) + G(x+ct)$

Conditions : $\left.\begin{array}{l} u(x,t)\big|_{t=0} = f(x) \\ u_t(x,t)\big|_{t=0} = g(x) \end{array}\right] \Rightarrow \left\{\begin{array}{l} F(x) = \frac{f(x)}{2} - \frac{1}{2c}\int_A^x d\xi\, g(\xi) + K/2 \\ G(x) = \frac{f(x)}{2} + \frac{1}{2c}\int_A^x d\xi\, g(\xi) - K/2 \end{array}\right.$

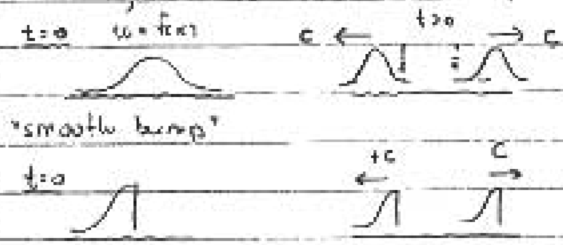
$$\boxed{u(x,t) = \underbrace{\frac{f(x+ct)+f(x-ct)}{2}}_{} + \underbrace{\frac{1}{2c}\int_{x-ct}^{x+ct} d\xi\, g(\xi)}_{}}$$ D'Alembert Solution

contribution from different parts of the solution

(i) Let's take $g(x) = 0 \Rightarrow u(x,t) = \dfrac{f(x+ct)+f(x-ct)}{2}$

Remark 1:

at $t=0$, we start with a bump:



- initial bump multiplied by $\frac{1}{2}$ splits to two
- one goes at c, the other at $-c$

! • discontinuities can propagate (in the Initial data)

because the system is hyperbolic, we have real CHAR...

Remark 2:

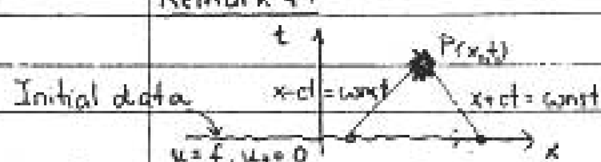
signals cannot propagate faster than c. (general)

In our case, they propagate exactly at $|c|$

Remark 3:

$f(x \pm ct)$: travelling waves [interesting concept to look at particular solutions of the PDE, convert it to ODE PDE ⇒ ODE]

Remark 4:



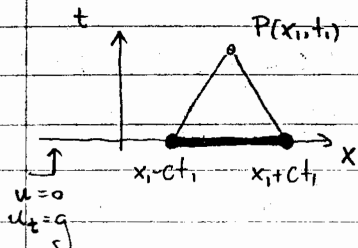
how the solution at P is affected by the data: draw CHAR

$\Rightarrow$ each particular point $P(x_1, t_1)$ is affected by the data at only two points $x_1 - ct_1$ and $x_1 + ct_1$

! # CHAR intersecting the data line and going through the point of interest = # of conditions

$\rightarrow$ very general feature of the hyperbolic equations

(ii) let's take $f(x) = 0 \Rightarrow u(x,t) = \frac{1}{2c} \int_{x-ct}^{x+ct} d\xi \, g(\xi)$

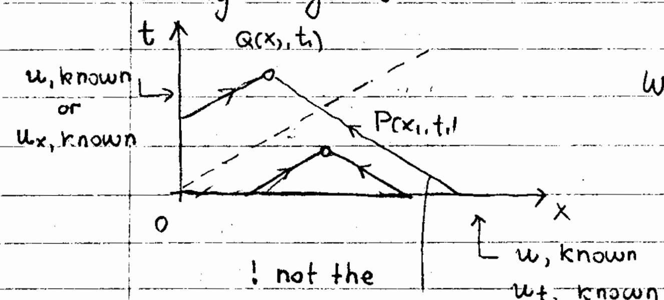


signal at $P(x_1, t_1)$ is affected by the data at all points between $x_1 - ct_1$, $x_1 + ct_1$

Signals are affected by portions of x-axis (initial data) that can reach (x_1, t_1) traveling not faster than $|c|$
call the segment)

The CHAR gives only the boundaries

Change regions of interest $x>0$, $t>0$ $\quad u_{tt} = c^2 u_{xx}$



What conditions do I need for a unique u?

CHAR = # condns on any axis
$\downarrow$
well posed for hyperbolic system

! not the other direction, it will violate causality (information wanted from earlier times!)

* Robin condition is right only for elliptic problems, but not for hyperbolic problems

* Interpret as a reflection.

Diffusion (or Heat) equation

$u_t = \nu \nabla^2 u$, $\nu > 0$ diffusivity

- describes temperature distribution
- wave propagation (low fr.) in conducting media (as seawater, approx. for Maxwell's equ.)

Ex 1D $\quad u_t = \nu \dfrac{\partial^2 u}{\partial x^2} \quad -\infty < x < \infty, \quad t > 0$

↓ it is a parabolic system

How to give conditions to this equation?

IC in time : $u(x, t=0) = f(x)$

$u(x \to \pm\infty, 0) = 0 \rightarrow$ goes to 0 "sufficiently fast"

i.e all derivatives go to 0 too

How to solve it? → FFT

Reasons: 1) The eq. is translation invariant ($x \to x+A$ unchanged)

2) The conditions in x are homogeneous

$u(x \to \pm\infty, t=0) = 0 \Rightarrow (\mathcal{L} u)\big|_{x \to \pm\infty} = 0$

Apply FFT:

def: Fourier tr. $\quad \tilde{u}(k,t) = \int_{-\infty}^{+\infty} dx\, u(x,t)\, e^{-ikx}$ / inverse: $u(x,t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} dk\, \tilde{u}(k,t)\, e^{ikx}$

what's the equation for the fourrier tr.?

$FT\{u_t\} = FT\{\nu u_{xx}\} \quad \Rightarrow \quad \boxed{\tilde{u}_t = -\nu k^2 \tilde{u}}$ → this is an ODE

integration by parts: $FT\{u_{xx}\} = \int_{-\infty}^{+\infty} dk\, \frac{\partial^2 u}{\partial x^2} e^{-ikx} = \cancelto{0}{\frac{\partial u}{\partial x} e^{-ikx}\Big|_{-\infty}^{+\infty}} + \cancelto{0}{iku\, e^{-ikx}\Big|_{-\infty}^{+\infty}}$

$- k^2 \int_{-\infty}^{+\infty} u\, e^{-ikx}\, dx$

After applying the FFT, the PDE goes to ODE (same theme)

$$\frac{d\tilde{u}}{dt} + \nu k^2 \tilde{u} = 0$$

$\Rightarrow \quad \tilde{u}(k,t) = A(k)\, e^{-\nu k^2 t}$

from initial condition: $\quad u(x, t=0) = f(x)$

$u(x,t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} dk\, A(k)\, e^{-\nu k^2 t}\, e^{ikx} \Big|_{t=0} \quad \Rightarrow \quad f(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} dk\, A(k)\, e^{ikx}$

$\Rightarrow \quad A(k) = FT\{f(x)\}$ $\quad A(k)$ is the Fourier transform of the IC $f(x)$

$A(k) = \int_{-\infty}^{+\infty} dx'\, f(x')\, e^{-ikx'}$ known

$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} dk \int_{-\infty}^{+\infty} dx'\, f(x')\, e^{-ix'k}\, e^{-\nu k^2 t}\, e^{ikx}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} dx'\, f(x') \left\{ \int_{-\infty}^{+\infty} dk\, e^{-\nu k^2 t + ik(x-x')} \right\}$$ can be calculated

$$\int_{-\infty}^{+\infty} dk\, e^{-\nu t\left[k^2 - \frac{ik(x-x')}{\nu t} - \frac{(x-x')^2}{4\nu^2 t^2}\right] - \nu t \frac{(x-x')^2}{4\nu^2 t^2}}$$

$$e^{-\frac{(x-x')^2}{4\nu t}} \int_{-\infty}^{+\infty} dk\, e^{-\nu t\left[k - \frac{x-x'}{2\nu t}\right]^2}$$

const, we can shift limits $\pm\infty$

$$e^{-\frac{(x-x')^2}{4\nu t}} \sqrt{\frac{\pi}{\nu t}} \qquad \left(\text{use } \int_{-\infty}^{+\infty} d\xi\, e^{-\xi^2} = \sqrt{\pi}\right)$$

$$\Rightarrow u(x,t) = \int_{-\infty}^{+\infty} dx'\, f(x') \underbrace{\left[\frac{1}{\sqrt{4\pi\nu t}}\, e^{-\frac{(x-x')^2}{4\nu t}}\right]}_{\text{"weight"}}$$

this tells how the IC influences the solution through a "weight" function.

Define: 1) $\delta(x) = 0$ for $x \neq 0$ — delta function

2) $\int_{-\varepsilon}^{\varepsilon} dx\, \delta(x) = 1$ any $\varepsilon > 0$

Property: $\int_{-\infty}^{+\infty} dx'\, f(x')\, \delta(x-x') = f(x)$

Take $f(x') = \delta(x') \Rightarrow u(x,t) = \sqrt{\frac{1}{4\pi\nu t}}\, e^{-\frac{x^2}{4\nu t}} = G(x,t)$

$x, t, f \Longrightarrow \boxed{\text{Diff eqn.}} \Longrightarrow u(x,t;f)$ Interpretation:

"Box"

$f(x) = \int_{-\infty}^{+\infty} dx'\, f(x')\, \delta(x-x') \Rightarrow u(x,t) = \int_{-\infty}^{+\infty} dx'\, f(x')\, \underline{G(x-x',t)}$

Green's function

$f(x) \to \boxed{} \to u$; $\delta(x-x') \to \boxed{} \to G$

converts PDE $\Rightarrow$ "integrals"

$\boxed{\text{Ex. 1}}$ $f(x) = \begin{cases} 1, & x>0 \\ 0, & x<0 \end{cases}$ $u(x,t) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{x/\sqrt{4\nu t}} d\xi\, e^{-\xi^2}$ (after a change of variables)

*) We have discontinuous Initial data? Can this discontinuity propagate?
NO, the integral gives us a smooth function (u and derivatives)
=> discontinuity data can NOT propagate via diff. system.

*) No sense of talking about finite speed of propagation in the solution. Solution appears to "propagate" everywhere instantaneously.
↳ don't really propagate, spread

spread, but the area is the same
because diffusion equation comes from conservation law

$$\begin{cases} \frac{\partial g}{\partial t} + \frac{\partial q}{\partial x} = 0 \\ q = -\nu g_x \end{cases}$$

March 8, 2004 Lecture 10

Opt. Reading Debnath 1.1-1.9
Rev. Session FRI, 4:15-5:45
Quiz 1 : MON, March 15

Diffusion eqn: IVP $\begin{cases} u_t = \nu u_{xx} & , -\infty < x < \infty,\ t>0 \\ u(x, t=0) = f(x) \\ u(x,t) \to 0 \text{ as } |x| \to \infty & \text{"sufficiently fast"} \end{cases}$
($u \to 0$, $u_x \to 0$, $u_{xx} \to 0$ etc.)

FT (in x) : Solution $u(x,t) = \int_{-\infty}^{+\infty} dx'\, f(x')\, \underbrace{\frac{1}{\sqrt{4\pi\nu t}}\, e^{-\frac{(x-x')^2}{4\nu t}}}$

$G(x,x';t,t') = \frac{1}{\sqrt{4\pi\nu t}}\, e^{-\frac{(x-x')^2}{4\nu(t-t')}}$, $t > t'$, Green's function $G(x,x';t,0)$ (Green's function is also useful for non-linear equations)

Feature: Area is conserved $\int_{-\infty}^{+\infty} dx\, u(x,t) = \text{const}$ Why?

① Diffusion eqn. comes from a conservation law, $g_t + q_x = 0$ ($g \leftrightarrow u$, $q = -\nu u_x$)