March 17, 2004 — Lecture 13

Pick up: Practice Set 3

Homework 3 to be posted TODAY

Opt. Reading
- Kevorkian 2.1 - 2.5
- Debnath 1.6
- Evans 2.1 - 2.4

Theme: PDE(s) $\rightarrow$ ODE(s)

I. Directly:

Characteristics, separation of variables

traveling waves
similarity solutions (later on) } "guess"

II. For nonlinear PDE

convert nonlinear PDE to linear PDE first

① Transformation $u \curvearrowright w = \phi(u)$

② Potential functions

③ Hodograph transform (switch independent and dependent variables)

④ Legendre transformation: view derivatives of unknown functions as independent variables

<u>Example</u>: $(1+u_y^2)\,u_{xx} - 2\underbrace{u_x u_y u_{xy}}_{\text{cubic term}} + (1+u_x^2)\,u_{yy} = 0$

but the eq. is QL, 2nd order, no explicit dependence on x, y.

- the derivatives differ by one and only two orders appear.

$\Rightarrow$ View u_x and u_y as independent variables

$$\begin{cases} p = u_x(x,y) \\ s = u_y(x,y) \end{cases} \quad \text{definition} \qquad (x,y) \curvearrowright (p,s)$$

assuming mapping is 1-1 and invertible

Condition: $\begin{vmatrix} u_{xx} & u_{xy} \\ u_{yx} & u_{yy} \end{vmatrix} \neq 0 \longrightarrow \begin{cases} x = x(p,s) \\ y = y(p,s) \end{cases}$

① Define: Legendre transform of u is function of p, s that has a 1-1 relation with u: $\underset{(x,y)}{u} \leftrightarrow \underset{(p,s)}{v}$

$$v(p,s) = px + sy - u(x,y) = p\,x(p,s) + s\,y(p,s) - u(x(p,s), y(p,s))$$

② Find a PDE for v with independent variables p, s
→ we need to find the second derivatives only

Exercise: (find)

$u_{xx} = \mathcal{J} v_{ss}$, $u_{xy} = -\mathcal{J} v_{ps}$, $u_{yy} = \mathcal{J} v_{ss}$

where

$$\mathcal{J} = \begin{vmatrix} u_{xx} & u_{xy} \\ u_{xy} & u_{yy} \end{vmatrix}$$ assuming $u_{xy} = u_{yx}$ (Hessian)

assumes $\mathcal{J} \neq 0$

Replace in the PDE for $u(x,y)$:

$$(1+s^2)\cancel{\mathcal{J}} v_{ss} + 2ps\,\cancel{\mathcal{J}} v_{ps} + (1+p^2) v_{pp}\cancel{\mathcal{J}} = 0$$

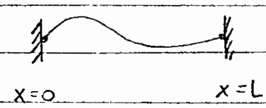
LINEAR PDE!

Method I

Anatomy of separation of variables (Ⅰ)

"Natural frequencies" or eigenvalue problem

Example: Vibrating string



$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 & x \in [0, L],\ t > 0 \\ u(0,t) = u(L,t) = 0 & \text{Boundary conditions} \\ u(x,0) = f(x) & \text{Initial conditions} \\ u_t(x,0) = 0 & \end{cases}$$

① go to a system but you will get stuck because of the BC.
→ separation of variables

what are the variables: x, t

② Try $u(x,t) = X(x)T(t) \Rightarrow X T''(t) - c^2 T X''(x) = 0$

$$\underbrace{\frac{T''}{T}(t)}_{\text{only on } t} - c^2 \underbrace{\frac{X''}{X}}_{\text{only on } x} = 0$$

③ Get ODE's $\dfrac{T''}{T} = -\lambda$, $\dfrac{X''}{X} = -\lambda/c^2$ say $\lambda > 0$

$\frac{T''}{T} = -\lambda c^2 \Rightarrow T(t) = A\cos(\sqrt{\lambda}t) + B\sin(\sqrt{\lambda}t)$ $\quad \sqrt{\lambda} = \omega$ frequencies

$\frac{X''}{X} = -\frac{\lambda}{c^2} \Rightarrow X(x) = C\cos\left(\frac{\sqrt{\lambda}}{c}t\right) + B\sin\left(\frac{\sqrt{\lambda}}{c}t\right)$ $\quad \frac{\sqrt{\lambda}}{c} = K$ wave number

④ Apply HOMOGENEOUS BC.

- $u(0,t) = u(L,t) = 0 \Rightarrow X(0) = X(L) = 0$

  1\) $\Rightarrow C = 0$

  2\) $\Rightarrow B\sin\left(\frac{\sqrt{\lambda}}{c}L\right) = 0 \Rightarrow \frac{L\sqrt{\lambda}}{c} = n\pi$, $n = \cancel{0}, 1, 2, \ldots$ ↑ trivial

  $\lambda = \left(\frac{n\pi c}{L}\right)^2 \Rightarrow \omega_n = \frac{n\pi c}{L}$

- $u_t(x,0) = 0 \Rightarrow T'(0) = 0 \Rightarrow B = 0$

Solution:

$u(x,t) = D\sin\left(\frac{n\pi}{L}x\right)\cos\left(\frac{n\pi c}{L}t\right)$ $\quad n = 1, 2, 3, \ldots$

So far we solved the eigenvalue problem: Find non-zero solutions of the PDE satisfying homogeneous conditions. This gave us discrete values for (ω, K)

$\omega = \omega_n$ are called "natural frequencies" of the system, arise naturally in seperation of variables

⑤ Apply the remaining condition

Supperimpose over all possible solutions:

$$u(x,t) = \sum_{n=1}^{+\infty} D_n \sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi c}{L}t\right)$$

$$u(x,0) = f(x) = \sum_{n=1}^{+\infty} D_n \sin\left(\frac{n\pi x}{L}\right) \underset{\text{Fourier series}}{\Rightarrow} D_n = \frac{2}{L}\int_0^L dx \sin\left(\frac{n\pi}{L}x\right) f(x)$$

Observations:

apply only

- "Linear" problems: the operator of the PDE is linear (we multiply the solution by const, satisfies same PDE)

2nd order case

- Homogeneous conditions $\Rightarrow$ we end up with an ODE

$$\boxed{\frac{d}{dx}\left[p(x)\frac{d}{dx}\right]\psi(x) + \left[q(x) + \mu s(x)\right]\psi(x) = 0} \qquad a < x < b$$

Sturm-Liouville problem

$+$ homogeneous BC at $x = a, b$ where $p, s > 0$, $q \leq 0$

The Sturm-Liouville problem has discrete set of solutions $\{\Psi_n\}$ corresponding to $\{\mu = \mu_n\}$.

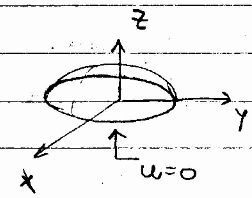
Under some restrictions, the set $\{\Psi_n\}$ is complete which means that any arbitrary function can be expanded over this set $\{\Psi_n\}$

$$f(x) = \sum_n c_n \Psi_n(x)$$

The set can be orthogonal $\int_a^b dx\, \underbrace{s(x)}_{\text{weight}}\, \Psi_n(x)\, \Psi_m(x) = 0$ if $m \neq n$

$$\Rightarrow \quad c_n = \frac{\int_a^b dx\, s(x)\, \Psi_n(x)\, f(x)}{\int_a^b dx\, s(x)\, \Psi_n(x)^2}$$

Example : Vibrating membrane



$u(x,y,t)$ circular membrane $r=a$

$$\begin{cases} u_{tt} - c^2 \nabla^2 u = 0 \\ u(r=a, \phi, t) = 0 \\ u(r, \phi, t=0) = f(r, \phi) \\ u_t(r, \phi, t=0) = 0 \quad \text{"zero" velocity} \end{cases}$$

Polar coordinates: (r, ϕ) + time t

Try separation of variables : $u(r,\phi,t) = \Lambda(r,\phi)\, T(t)$

Apply the PDE :

$$\underbrace{\frac{T''}{T}} - \underbrace{\frac{c^2 \nabla^2 \Lambda}{\Lambda}} = 0 \quad \Rightarrow \begin{cases} \frac{T''}{T} = -\lambda \\ c^2 \frac{\nabla^2 \Lambda}{\Lambda} = -\lambda \end{cases}$$

What is the sign of $(-\lambda)$?

$\Lambda|_{\text{boundary}} = 0$ use this $\Rightarrow$ $c^2 \nabla^2 \Lambda = -\lambda \Lambda$

$$c^2 \Lambda \nabla^2 \Lambda = -\lambda \Lambda^2$$

$$\int_{\text{disk}} c^2 \Lambda \nabla^2 \Lambda = \int_{\text{disk}} -\lambda \Lambda^2$$

$$\cancel{c^2 \oint_{\partial \text{disk}} \Lambda \nabla \Lambda . \underline{n}} - c^2 \int_{\text{disk}} |\nabla \Lambda|^2 dS = -\lambda \int_{\text{disk}} \Lambda^2 dS \quad \text{(because of homogeneous BC's)}$$

$$0 \quad \Rightarrow \quad c^2 \int_{\text{disk}} dS\, |\nabla \Lambda|^2 = \lambda \int_{\text{disk}} \Lambda^2 dS \quad \Rightarrow \quad \lambda > 0$$

variational formula for λ

$$\boxed{\lambda = c^2 \frac{\int dS\, |\nabla \Lambda|^2}{\int dS\, \Lambda^2} > 0}$$