18.306 : Advanced PDEs with Applications

Handout 6 : Review of Fourier series and Fourier transforms

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Generalities:

(A) Fourier series (classical theory)

Recall that a "class" of functions that are periodic in $(-\infty,\infty]$ with period $b-a$ admit a series expansion in sines and cosines :

$$f(x) = \sum_{n=0}^{\infty} a_n \cdot \cos\left(\frac{2n\pi x}{b-a}\right) + \sum_{n=1}^{\infty} b_n \cdot \sin\left(\frac{2n\pi x}{b-a}\right), \qquad a<x<b,$$

where

$$a_0 = \frac{1}{b-a}\int_a^b dx \cdot f(x), \qquad a_{n\neq 0} = \frac{2}{b-a}\int_a^b dx \cdot f(x)\cos\left(\frac{2n\pi x}{b-a}\right),$$

$$b_n = \frac{2}{b-a}\int_a^b dx \cdot f(x)\cdot \sin\left(\frac{2n\pi x}{b-a}\right). \qquad (b_0=0).$$

This expansion holds if $f(x)$ is square integrable in $[a,b]$:

$$\int_a^b dx \cdot |f(x)|^2 < \infty .$$

The convergence of the aforementioned series is then interpreted as "convergence in the mean," i.e.,

$$\lim_{N\to\infty} \int_a^b dx \cdot \left| f(x) - \sum_{n=0}^{N}\left[a_n \cdot \cos\left(\frac{2n\pi x}{b-a}\right) + b_n \cdot \sin\left(\frac{2n\pi x}{b-a}\right)\right]\right|^2 = 0.$$

This formula means that the RHS of the expansion for $f(x)$ may not converge to $f(x)$ pointwise, and hence is of different nature from, say, the Taylor expansion.

In particular, for the symmetric interval $[-L, L]$, where $a=-L$ and $b=L$, one gets

$$f(x) = \sum_{n=0}^{\infty}\left[a_n \cdot \cos\left(\frac{n\pi x}{L}\right) + b_n \cdot \sin\left(\frac{n\pi x}{L}\right)\right],$$

where

$$a_0 = \frac{1}{2L}\int_a^b dx \cdot f(x), \qquad \left.\begin{matrix} a_{n\neq 0} \\ b_n \end{matrix}\right\} = \frac{1}{L}\int_a^b dx \cdot f(x) \cdot \left\{\begin{matrix} \cos\left(\frac{n\pi x}{L}\right) \\ \sin\left(\frac{n\pi x}{L}\right) \end{matrix}\right. .$$

The last expression for $f(x)$ can be recast in an elegant form by replacing the sines and cosines by exponentials:

$$\cos\left(\frac{n\pi x}{L}\right) = \frac{e^{in\pi x/L} + e^{-in\pi x/L}}{2}, \quad \sin\left(\frac{n\pi x}{L}\right) = \frac{e^{in\pi x/L} - e^{-in\pi x/L}}{2i}.$$

Accordingly,

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left(\frac{a_n - ib_n}{2}\right) \cdot e^{i\frac{n\pi x}{L}} + \sum_{n=1}^{\infty}\left(\frac{a_n + ib_n}{2}\right) \cdot e^{-i\frac{n\pi x}{L}}$$

$$= \sum_{n=-\infty}^{\infty} c_n \, e^{in\pi x/L},$$

by changing the summation variable from n to $-n$ in the second sum. In the above,

$$c_n = \begin{cases} a_0, & n=0 \\ \dfrac{a_{|n|} \mp i\, b_{|n|}}{2}, & n \gtrless 0. \end{cases}$$

$$\Rightarrow \quad c_n = \frac{1}{2L}\int_{-L}^{L} dx \cdot f(x)\, e^{-in\pi x/L}, \qquad \text{all } n.$$

(B) Fourier transform

We start with the Fourier series

$$f(x) = \sum_{n=-\infty}^{\infty} c_n \, e^{in\pi x/L} \quad , \quad x \text{ in } (-L, L),$$

and we eventually allow $L \to \infty$ in order to find a representation for $f(x)$ in $(-\infty, \infty)$. For this purpose, consider the quantity

$$\omega_n = \frac{n\pi}{L} \quad \Rightarrow \Delta\omega_n \equiv \omega_{n+1} - \omega_n = \frac{\pi}{L} \longrightarrow 0 \quad \text{as} \quad L \to +\infty.$$

Hence, in the limit $L \to \infty$ ω_n can be treated as a continuous variable.

The coefficients c_n are

$$c_n = \frac{1}{2L} \int_{-L}^{L} dx \; f(x) \; e^{-in\pi x/L}$$

$$\rightarrow \quad (c_n \cdot 2L) = \int_{-L}^{L} dx \; f(x) \; e^{-i\omega_n x}$$

If we <u>assume</u> that the integral in RHS <u>converges</u> as $L \to \infty$, then (by treating $\omega_n \equiv \omega$ as a <u>continuous</u> variable) we get

$$\lim_{L\to\infty} (c_n \cdot 2L) = \int_{-\infty}^{\infty} dx \; f(x) \, e^{-i\omega x} \equiv \tilde{f}(\omega) \; : \; \text{Fourier transform (FT) of } f(x) \text{ in } (-\infty, \infty)$$

The function $f(x)$ is recovered as

$$f(x) = \sum_{n=-\infty}^{\infty} c_n \, e^{in\pi x/L} = \sum_{n=-\infty}^{\infty} (c_n \cdot 2L) \frac{1}{2L} \, e^{in\pi x/L} = \sum_{n=-\infty}^{\infty} \frac{\tilde{f}(\omega_n)}{2L} \, e^{i\omega_n x}$$

$$= \sum_{n=-\infty}^{\infty} \tilde{f}(\omega_n) \cdot e^{i\omega_n x} \cdot \left(\frac{\Delta\omega_n}{2\pi}\right) \xrightarrow[L\to\infty]{} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \, \tilde{f}(\omega) \, e^{i\omega x} \; : \; \text{"inverse F.T."}$$

Particulars of the Fourier transform:

In most part of this course we define the F.T. of a function $f(x)$, where $-\infty < x < \infty$, as

$$\tilde{f}(\mathfrak{J}) = \int_{-\infty}^{\infty} dx \; f(x)\, e^{-i\mathfrak{J}x}. \tag{1}$$

(I will use mostly $\mathfrak{J}$ instead of ω as the Fourier variable).
There are two distinct theories of Fourier transforms: (Now you must forget how we derived the formulas for the FT on p. 3 since the discussion there was heuristic.)

(I) Fourier transform of square integrable functions

It is assumed that $$\int_{-\infty}^{\infty} dx \; |f(x)|^2 < \infty. \tag{2}$$

The inverse F.T for (1) is given by

$$f(x) = \int_{-\infty}^{\infty} \frac{d\mathfrak{J}}{2\pi}\, e^{i\mathfrak{J}x}\, \tilde{f}(\mathfrak{J}). \qquad [\text{Note: Don't forget the } (2\pi)^{-1}.] \tag{3}$$

Note that in this case $\tilde{f}(\mathfrak{J})$ is defined for real $\mathfrak{J}$. Accordingly, the path of integration in (3) coincides with the entire real axis, i.e., all $\mathfrak{J}$'s – over which we integrate – are real.
Equation (1) is meaningful in the sense of "convergence in the mean," i.e., (1) means that there exists $\tilde{f}(\mathfrak{J})$ for all real $\mathfrak{J}$ such that

$$\lim_{R\to+\infty} \int_{-\infty}^{\infty} d\mathfrak{J} \cdot \left| \tilde{f}(\mathfrak{J}) - \int_{-R}^{R} dx \; f(x)\, e^{-i\mathfrak{J}x} \right|^2 = 0. \tag{4}$$

Symbolically, one writes
$$\tilde{f}(\vartheta) = \underset{R\to+\infty}{\text{l.i.m.}} \int_{-R}^{R} dx \quad f(x)\ e^{-i\vartheta x}. \tag{5}$$

Similarly, Eq. (3) means that, given $\tilde{f}(\vartheta)$, there exists an $f(x)$ such that
$$\lim_{R\to+\infty} \int_{-\infty}^{+\infty} dx \cdot \left| f(x) - \int_{-R}^{R} \frac{d\vartheta}{2\pi}\ e^{i\vartheta x}\ \tilde{f}(\vartheta) \right|^2 = 0. \tag{6}$$

It can then be proved that
$$\int_{-\infty}^{\infty} d\vartheta\ |\tilde{f}(\vartheta)|^2 = 2\pi \int_{-\infty}^{+\infty} dx\ |f(x)|^2, \tag{7}$$

which is Parseval's identity for square integrable functions.
We see that the pair $(f(x), \tilde{f}(\vartheta))$ defined this way consists of two functions with very similar properties. This situation may change drastically if condition (2) is relaxed.

(II) Fourier transforms of integrable functions

The condition on $f(x)$ reads as
$$\int_{-\infty}^{+\infty} dx \cdot |f(x)| < \infty. \tag{8}$$

Then $\tilde{f}(\vartheta)$ is still defined for real ϑ. Indeed, from (1) one gets
$$|\tilde{f}(\vartheta\text{:real})| = \left| \int_{-\infty}^{+\infty} dx \quad f(x)\, e^{-i\vartheta x} \right| \le \int_{-\infty}^{+\infty} dx \quad |f(x) \cdot e^{-i\vartheta x}| = \int_{-\infty}^{+\infty} dx \cdot |f(x)| < \infty. \tag{9}$$

One can further show that the function
$$\tilde{f}_{+}(\vartheta) = \int_{-\infty}^{0} dx \quad e^{-i\vartheta x} \quad f(x) \tag{10}$$

is analytic in the upper half of the ζ plane, while

$$\tilde{f}_+(\zeta) \to 0 \quad \text{as} \quad |\zeta| \to \infty \quad \text{with} \quad \text{Im}\,\zeta > 0. \tag{11}$$

Similarly, the function

$$\tilde{f}_-(\zeta) = \int_0^\infty dx \; e^{-i\zeta x} \; f(x) \tag{12}$$

is analytic in the lower half plane and

$$\tilde{f}_-(\zeta) \to 0 \quad \text{as} \quad |\zeta| \to \infty \quad \text{with } \text{Im}\,\zeta < 0.$$

Clearly,

$$\tilde{f}(\zeta) = \tilde{f}_+(\zeta) + \tilde{f}_-(\zeta) \quad , \quad \zeta : \text{real}. \tag{13}$$

It can be shown that

$$\tilde{f}(\zeta) \to 0 \quad \text{as} \quad \zeta \to \pm\infty \quad (\zeta : \text{real}), \tag{14}$$

a property in common with FT's of square integrable functions.

Example Solve the boundary-value problem $\begin{cases} u''(x) - \lambda^2 u(x) = f(x) \\ u(x \to \pm\infty) = 0 \;, \; -\infty < x < \infty \end{cases}$ $(\lambda > 0)$ ("sufficiently fast")

by use of the Fourier transform.

Solution: The FT of $u(x)$ is (I use $k \equiv \zeta$)

$$\tilde{u}(k) = \int_{-\infty}^{\infty} dx \; e^{-ikx} \; u(x) \tag{15}$$

By taking the FT of both sides of the given ODE, we obtain

$$\underbrace{\int_{-\infty}^{\infty} dx\ e^{-ikx}\ u''(x)}_{\text{(integration by parts)}} - \lambda^2 \underbrace{\int_{-\infty}^{\infty} dx\ e^{-ikx}\ u(x)}_{\tilde{u}(k)} = \underbrace{\int_{-\infty}^{\infty} dx\ e^{-ikx}\ f(x)}_{\tilde{f}(k):\ \text{FT of } f(x)\ \text{(given)}}$$

$$\left\{ \begin{array}{l} e^{-ikx} u'(x) \Big|_{x=-\infty}^{\infty} + ik\ e^{-ikx} u(x) \Big|_{x=-\infty}^{\infty} \\ - k^2 \displaystyle\int_{-\infty}^{\infty} dx\ e^{-ikx}\ u(x) \end{array} \right.$$

By assuming that $u(x \to \pm\infty) = 0$ <u>and</u> $u'(x \to \pm\infty) = 0$ the last equations give an algebraic equation for $\tilde{u}(k)$:

$$-(k^2 + \lambda^2)\ \tilde{u}(k) = \tilde{f}(k) \quad \Rightarrow \quad \tilde{u}(k) = - \frac{\tilde{f}(k)}{k^2 + \lambda^2} \qquad (16)$$

Hence,

$$u(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk\ e^{ikx}\ \tilde{u}(k) = \frac{-1}{2\pi} \int_{-\infty}^{\infty} dk\ e^{ikx}\ \frac{\tilde{f}(k)}{k^2 + \lambda^2}$$

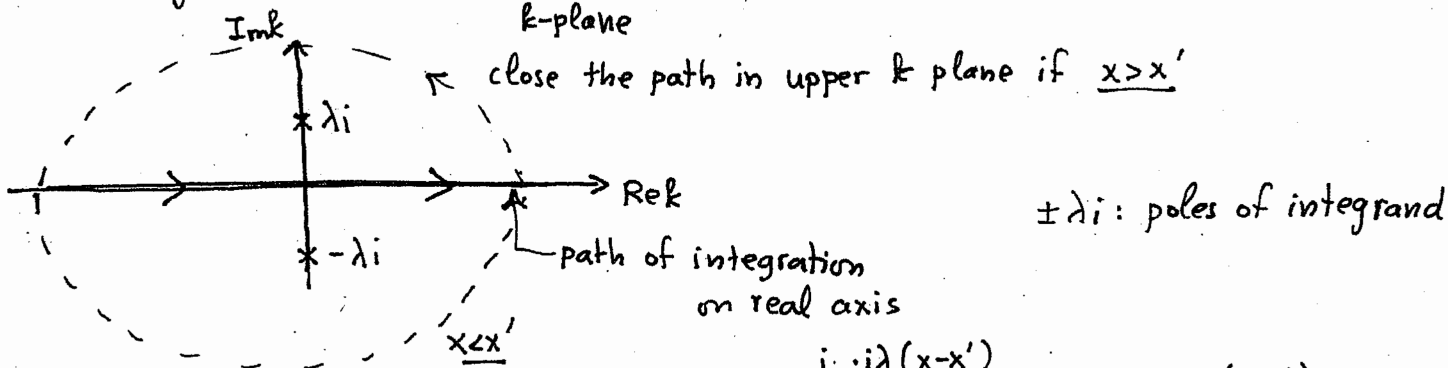
We may further simplify this formula using

$$\tilde{f}(k) = \int_{-\infty}^{\infty} dx'\ e^{-ikx'}\ f(x').$$

Then,

$$u(x) = \frac{-1}{2\pi} \int_{-\infty}^{\infty} dx'\ f(x') \int_{-\infty}^{\infty} dk\ \frac{e^{ik(x-x')}}{k^2 + \lambda^2} \qquad (17)$$

The integral in k can be evaluated by various means (for example, by contour integration, as described below).



$$\int_{-\infty}^{\infty} dk \, \frac{e^{ik(x-x')}}{k^2+\lambda^2} = \begin{cases} \underline{x>x'}: \; 2\pi i \cdot \dfrac{e^{i\cdot i\lambda(x-x')}}{2i\lambda} = \dfrac{\pi}{\lambda} \, e^{-\lambda(x-x')} \\[2ex] \underline{x<x'}: \; -2\pi i \, \dfrac{e^{i(-i\lambda)\cdot(x-x')}}{-2i\lambda} = \dfrac{\pi}{\lambda} \, e^{\lambda(x-x')} \end{cases} \tag{18}$$

via applying the Cauchy integral formula.

Finally, from (17)

$$u(x) = -\frac{1}{2\lambda} \int_{-\infty}^{\infty} dx' \, f(x') \, e^{-\lambda\cdot|x-x'|} . \tag{19}$$