

The description of the references given in parentheses can be found in the Bibliography for 18.306.

8. This problem concerns the case of a non-homogeneous PDE. Find an infinite series expansion for  $u(x, y)$  in the strip  $x \geq 0, 0 \leq y \leq \pi$ , where  $u$  satisfies the PDE  $u_{xx} + 2u_x + u_{yy} = 1$ , and the boundary conditions  $u(x, 0) = u_y(x, \pi) = 0, u(0, y) = 0$  and  $u(\infty, y)$ : finite. **Hint:** Expand  $u$  in Fourier series in  $y, u = \sum_n [a_n(x) \sin(k_n y) + b_n(x) \cos(k_n y)]$  where  $k_n$  are constants, and first find  $b_n(x)$  and  $k_n$  so that the given conditions at  $y = 0, \pi$  are satisfied. Then find  $a_n(x)$  via suitable equation(s).
9. (Evans, Prob. 4.7.3(i), p. 234.) This problem involves the conversion of a nonlinear PDE to an ODE *directly*, i.e., via the *assumption* (or “guessing”) of a solution with a particular form. Consider the viscous conservation law  $u_t + [F(u)]_x - au_{xx} = 0$  where  $u = u(x, t)$ ,  $a$  is a positive constant ( $a > 0$ ), and  $F(u)$  is a given, sufficiently well-behaved function of  $u$ . Find a particular solution of this PDE in the form of a traveling wave,  $u(x, t) = v(x - \sigma t)$  where  $\sigma$  is a constant speed. **Hint:** Denoting  $\xi = x - \sigma t$ , find an ODE in  $\xi$ , and *solve* it.
10. Consider the system of *nonlinear* PDEs (1):

$$u^2 u_x - uv(u_y + v_x) + v^2 v_y = 0, \quad u_y - v_x = 0, \quad (1)$$

where  $u = u(x, y)$  and  $v = v(x, y)$  are unknown functions.

- (a) Convert (1) to a system of linear PDEs.
- (b) Find an equivalent single PDE by using a potential function,  $\Omega$ . Find particular (nonzero) solutions of this PDE by applying separation of variables. **Hint:** The problem does not give any conditions. So, give your solution in terms of arbitrary integration constants. Why is that solution a particular one?
11. A membrane of uniform mass per unit area  $m$  has the shape of the circular sector  $0 \leq \phi \leq \pi/2, 0 < a \leq r \leq b$  ( $r, \phi$ : polar coordinates), and is subjected to tension  $S$ . The displacement  $u(r, \phi, t)$  is zero on the boundary  $r = a, r = b, \phi = 0, \phi = \pi/2$ . Derive an equation that describes the natural frequencies of the membrane. **Hint:** Take the PDE  $\nabla^2 u = (m/S)u_{tt}$  and eliminate time by setting  $u = v(r, \phi)e^{-i\omega t}$ .