Set $\varepsilon = 0$: $u_x = \frac{4y^2}{1-y^2}$ $\Rightarrow$ $u = \frac{4xy^2}{1-y^2} + c(y)$

no BL at $x=0$ so $u(x=0,y) = 0$ $\Rightarrow$ $C(y) = 0$

$h(x,y) \xrightarrow[y \to -1]{} \infty$! not finite value

May 12, 2004 Lecture 28

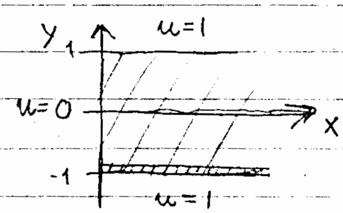
Opt. Redding on Solitons:

- Debnath 9.2–9.6, 11.7, 11.8
- Whitham 13.10–13.12
- Drazin & Johnson, Chs. 1.2

Pick up: Graded Hmwk 5

Boundary layers theory

Example : $\varepsilon u_{yy} - (1-y^2)u_x + 4y^2 = 0$ $u = u(x,y)$



Symmetry : $u(x,y) = u(x,-y)$

I $\varepsilon = 0$:

$u_x = \frac{4y^2}{1-y^2}$ $\Rightarrow$ $u(x,y) = h(x,y) = \frac{4xy^2}{1-y^2} + C(y)$

no boundary layer at $x=0$ $\Rightarrow$ $h(x=0,y) = 0$ $\Rightarrow$ $C(y) = 0$

$\Rightarrow$ $h(x,y) = \frac{4xy^2}{1-y^2}$! when $y \to \pm 1$ $h(x,y) \to \infty$
outer solution blows up!

Does not satisfy conditions at $y = \pm 1$ $\Rightarrow$ boundary layers?

(i) $y = -1$: $\eta = \frac{1+y}{\varepsilon^\alpha}$, $\alpha > 0$ to be found

$\eta = O(1)$ in the boundary layer : $u(x,y;\varepsilon) = h(x,y) + p(x,\eta)$

PDE for p:

$\varepsilon^{1-2\alpha} p_{\zeta\zeta} + \varepsilon h_{yy} - (1-y^2) p_x = 0 \quad + \quad (1-y^2) h_x = 4y^2$

inside the boundary layer : $\zeta = O(1)$, $p_{\zeta\zeta} = O(1)$, $\underbrace{h_{yy} = O(1)}_{\text{not true!}}$, $p_x = O(1)$

How it blows up?

since h_{yy} blows up as $y \to -1$ inside the layer

$$h(x,y) = \frac{4x(-1+\zeta\varepsilon^{\alpha})^2}{\underbrace{1-(-1+\zeta\varepsilon^{\alpha})^2}_{(2-\varepsilon^{\alpha}\zeta)(\varepsilon^{\alpha}\zeta)}} \underset{\substack{\zeta = O(1)\\ |\varepsilon| \ll 1}}{\approx} \frac{4x}{2\zeta} \boxed{\varepsilon^{-\alpha}}$$

$\uparrow$ $y = -1 + \zeta\varepsilon^{\alpha}$

$h(x,y) = O(\varepsilon^{-\alpha})$

$\Rightarrow$ εh_{yy} CAN NOT be neglected

$$p_{\zeta\zeta} + \varepsilon^{2\alpha} h_{yy} - \varepsilon^{2\alpha-1}(1-y^2) p_x = 0$$

usually we neglect $\varepsilon^{2\alpha} h_{yy}$ and find $\alpha = \frac{1}{2}$

$\Rightarrow$ can not do this here

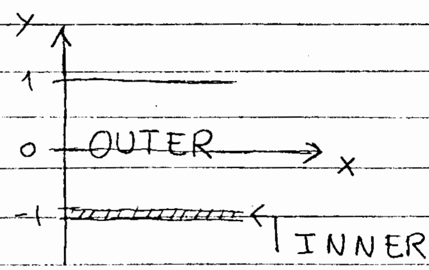
because $h_{yy} \neq O(1)$

Difficult point: outer solution scales as negative power of ε.

Outer solution:

$$h(x,y) = \frac{4xy^2}{1-y^2}$$

$-1 < y < 1$ & $\zeta \geq O(1)$



Inner solution: $u_{in}(x,y)$

Overlap: we let $y \to -1$ in the outer solution

$$h(x,y) \underset{y \sim -1}{\longrightarrow} \frac{4x}{2(1+y)} = \frac{2x}{\zeta} \varepsilon^{-\alpha}$$

$\Rightarrow$ set $u_{in}(x,y) = \varepsilon^{-\alpha} \Psi(x,\zeta)$ where Ψ and derivatives are $O(1)$

For boundary layer problems it is advisible to separate inner and outer solution and then match them afterwards.

Composite approach : $u = h + p$ (outer: h, inner: p)
works only if h & p are of the same order

Boundary layer $\gamma = O(1)$, find PDE for Ψ

$$\varepsilon u_{in,yy} - (1-y)\, u_{in,x} + yy^2 = 0$$

$$\Rightarrow \underbrace{\varepsilon^{1-3\alpha}}_{1} \Psi_{\gamma\gamma} - 2\gamma \Psi_x + y = 0$$

Simplifications:
$y \approx -1$
$1-y^2 = 2\varepsilon^{\alpha}\gamma$
$(1-y)(1+y)$

$$\Psi_{\gamma\gamma} - 2\varepsilon^{3\alpha-1}\gamma\Psi_x + 4\varepsilon^{3\alpha-1} = 0$$

assuming $\Psi_{\gamma\gamma}, \Psi_x, \gamma = O(1)$

in order to get an eq. of order $1 \Rightarrow 3\alpha = 1$ $\boxed{\alpha = \frac{1}{3}}$

PDE for Ψ: $\boxed{\Psi_{\gamma\gamma} - 2\gamma\Psi_x + y = 0}$

Conditions:

$$u_{in}(x,y) = \varepsilon^{-\alpha}\Psi(x,\gamma) \Rightarrow \Psi(x,\gamma) = \varepsilon^{1/3} u_{in}(x,y)$$

At $y=-1$:

$$\Psi(x,\gamma) = \varepsilon^{1/3} u_{in}\big|_{y=-1} = \varepsilon^{1/3}\cdot 1 \approx 0$$

because we want $\Psi = O(1)$
no ε-dependence ..

$$\Rightarrow \boxed{\Psi(x,\gamma=0) = 0} \quad ①$$

At $x=0$:

$$u_{in}(x,y)\big|_{x=0} = 0 \Rightarrow \boxed{\Psi(x=0,\gamma) = 0} \quad ②$$

Last condition comes from matching: (outer and inner)

$u_{out} \equiv h(x,y)$, $u_{in}(x,y) = \varepsilon^{-\alpha}\,\Psi(x,\zeta)$

$\downarrow y \to -1$ $\qquad\qquad \downarrow \zeta \to +\infty$

$\boxed{\frac{2x}{\zeta}}\,\varepsilon^{-\alpha} \qquad\qquad \boxed{\Psi(x,\zeta\to+\infty)}\,\varepsilon^{-\alpha}$

$$\Rightarrow \boxed{\Psi(x,\zeta\to+\infty) \simeq \frac{2x}{\zeta}}$$

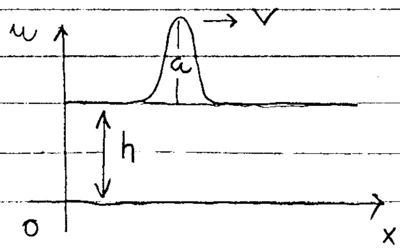
leading order condition for Ψ

asymptotically equal

Safe procedure: for singular perturbation theory treat inner and outer solution separetly.

Solitons:

Introduction: Historical origin John Scott Russell (1834) observed solitary wave: propagates without changing shape in Glasgow ~~River~~ Canal



$V^2 = g(h+a)$

- amplitude depends on speed
- does not change with time (long time) form

John Scott Russell was naturalist. = did not know any math.

Korteweg and de Vires (1895): wrote down a PDE for $\eta(x,t)$

KdV eqn. $\eta_t + c\left(1+\frac{3}{2}\frac{\eta}{h}\right)\eta_x + \underbrace{\left(\frac{1}{6}ch^2\right)}_{\gamma}\eta_{xxx} = 0$

h, c = const
↳ char. speed

Non-dimensional PDE: $\boxed{u_t + 6uu_x + u_{xxx} = 0}$

Standart KdV

Assume travelling wave:

$u(x,t) = \underbrace{f(x-Vt)}_{\xi} \Rightarrow$ find ODE for $f(\xi)$

KdV eqn.: $-Vf' + \underbrace{6ff'}_{(3f^2)'} + f''' = 0$

$\Rightarrow \quad -Vf + 3f^2 + f'' = \text{const} = A \quad | f'$

$-\frac{1}{2}Vf^2 + f^3 + \frac{1}{2}f'^2 = Af + B$

Take $f, f' \to 0$ when $|\xi| \to \infty$

$\Rightarrow B = 0$

divide by f $\quad -\frac{1}{2}Vf + f^2 + \frac{1}{2}\frac{f'^2}{f} = A$

extra assumption $\frac{f'^2}{f} \to 0 \quad \Rightarrow \quad A = 0$

First order ODE for f: $\quad f' = \pm f(-2f+V)^{1/2}$

$\Rightarrow \int \frac{df}{f(-2f+V)^{1/2}} = \pm \int d\xi + K$

$\downarrow$ set $f = \frac{V}{2}\frac{1}{\cosh^2\theta}$

$\hookrightarrow$ new variable

$\Rightarrow -\frac{2}{\sqrt{V}} \underbrace{\int d\theta}_{\theta} = \pm \xi + \underset{x_0}{\underset{\downarrow}{K}}$

solitary wave

$\downarrow$

$$\boxed{f(\xi) = \frac{V}{2} \frac{1}{\cosh^2\left[\frac{\sqrt{V}}{2}(x - Vt - x_0)\right]}}$$

$\hookrightarrow$ arbitrary

Other example: Sine-Gordon PDE

$$u_{xx} - u_{tt} = \sin u$$

- currents in Josephson-junction transmission lines (u_t: voltage)
- dislocation of crystals ($\sin u$: from periodic structure)
- waves in ferromagnets
- fields of laser pulses in non-linear media

Traveling wave: $u(x,t) = \phi(\underbrace{x - Vt}_{\xi})$

PDE for u $\Rightarrow$ $\frac{1}{2}(\phi')^2 + \frac{\cos\phi}{1-V^2} = B = \text{const}$

$$\Rightarrow \int_{\phi_0}^{\phi} \frac{d\psi}{\sqrt{A - \cos\psi}} = \pm\sqrt{\frac{2}{1-V^2}}\,(\xi - \xi_0)$$ where $\phi = \phi_0$ at $\xi = \xi_0$,

A arbitrary constant

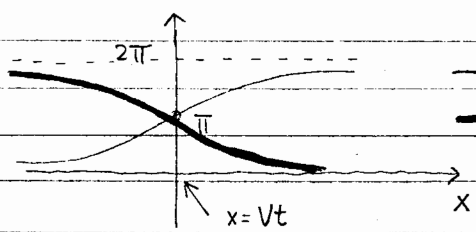
The value of A depends on V

If $A = 1$ $\rightarrow$ wave exists if $0 < V < 1$

integral can be find exactly

solution with +: $u = \phi(\xi) = 4 \arctan\left(e^{\frac{x-Vt}{\sqrt{1-V^2}}}\right)$

$\phi_0 = \pi$ at $\xi_0 = 0$



— "kink" solution

▬ "anti-kink" solution

solution with −: $u(x,t) = 4 \arctan\left(e^{\frac{x-Vt}{\sqrt{1-V^2}}}\right)$ $\phi_0 = \pi$ @ $\xi = 0$

"Inverse Scattering transform": Method to obtain "many soliton" solutions from certain PDE's