March 9, 2004 Lecture 11

Quiz 1 Take home Mon. March 15 4 hrs

To be picked 4-5 pm

Rev session: FRI 4:15-5:45 pm

Nonlinear PDE $\Rightarrow$ Linear PDE

Case: $u_t - a\nabla^2 u + b|\nabla u|^2 = 0 \quad t>0 \quad \vec{r} \in \mathbb{R}^n$

$u(r, t=0) = g(r)$ + condns at ∞

Steps ① Take $w = \phi(u)$; find ϕ s.t. PDE for w is linear

$\Rightarrow a\phi''(u) + b\phi'(u) = 0$ ODE

$\Rightarrow \phi(u) = \underbrace{c_1 e^{-\frac{bu}{a}} + c_2}_{\text{arbitrary}} = w$ (last time we took $\phi = e^{-\frac{bu}{a}}$)

② Choose c_1 and c_2 to accomodate boundary conditions

Example: Suppose $u \to 0$, $|\vec{r}| \to \infty$ and want $w \to 0$, $|\vec{r}| \to \infty$

then you choose $c_1 = -c_2$

③ Solve PDE for w: $\begin{cases} w_t = a\nabla^2 w \\ w(\vec{r}, t=0) = \phi(u(\vec{r}, t=0)) = \phi(g(\vec{r})) \\ w \to 0, \ |\vec{r}| \to \infty \end{cases}$

Let $c_1 = -1 = -c_2$, $\phi(u) = 1 - e^{-\frac{bu}{a}} = w$

in this way, we obtain homogeneous condition, we can apply directly our FT method

$$w(\vec{r}, t) = \int d\vec{r}'\, e^{-\frac{|\vec{r}-\vec{r}'|^2}{4at}} \frac{1}{(4\pi a t)^{n/2}} \left[1 - e^{-\frac{bg(\vec{r}')}{a}}\right]$$

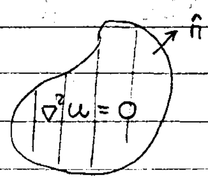
$$u(\vec{r}, t) = -\frac{a}{b} \ln(1 - w)$$

(for fixed problem in u, no matter how you play with the transformation and/or const c_1/c_2, you must have the same result)

Laplace equation $\nabla^2 u = 0$ $\left\{\begin{array}{l}\text{electro- magneto-statics}\\ \text{steady state diffusion}\\ \text{irrotational fluid flow}\end{array}\right.$

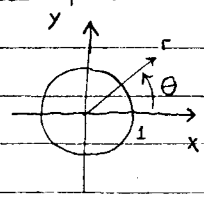
$\sim$ diffusion equation by setting $\frac{\partial}{\partial t} = 0$
we expect some similarities
Laplace = steady state of diffusion

"Natural" BVP : which means well-posed problems



$au + b\frac{\partial u}{\partial n} = c$ a, b, c given
$a \neq 0$ (for uniqueness)

Example:



2D POL (r, θ) , $0 \le r \le 1$, $0 \le \theta < \pi$
Laplace equation:

$$\left\{\begin{array}{ll}\nabla^2 u = 0 , & 0 \le r < 1 , \; 0 \le \theta < 2\pi\\ u(1,\theta) = f(\theta) & \end{array}\right.$$

(this is a well-posed problem)

(open set is a better idea)

Assumptions:
- u single valued
- $u(0,\theta)$ finite

Step 1 : Identify geometry, independent variables (r, θ)

Laplacian in polar coordinates: $\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$

Step 2 Apply PDE :

$$\Big(\underbrace{r^2\frac{\partial^2}{\partial r^2} + r\frac{\partial}{\partial r}}_{\mathcal{L}_1(r)} + \underbrace{\frac{\partial^2}{\partial \theta^2}}_{\mathcal{L}_2(r)}\Big) u = 0$$

depends only on r depends only on θ

$\rightarrow$ Apply separation of variables : Try $u(r,\theta) = R(r)\,\Theta(\theta)$
$\llcorner$ (multiplicative / additive)

$$\Rightarrow \underbrace{\frac{1}{R}\left[r^2 R''(r) + rR'(r)\right]}_{\text{depends only on } r} + \underbrace{\frac{\Theta''(\theta)}{\Theta(\theta)}}_{\text{depends only on } \theta} = 0$$

only way to have the sum equals 0 is each one to be const.

$$\Rightarrow \begin{cases} \frac{\Theta''}{\Theta} = -d = \text{const} & \text{I} \\ r^2R'' + rR' - dR = 0 & \text{II} \end{cases}$$

two ODEs

PDE $\Rightarrow$ ODE

I $\Theta'' + d\,\Theta''(\theta) = 0 \qquad 0 \leq \theta < 2\pi$

+ BC: usual assumption is that the solution u is single-valued $\Rightarrow$ therefore the BC are periodic $\circledast$ "physical" assumption

$$\begin{cases} \Theta(0) = \Theta(2\pi) \\ \Theta'(0) = \Theta'(2\pi) \end{cases}$$ means periodicity

the solution is $\Theta(\theta) = A_\pm e^{\pm i\sqrt{d}\theta} \xrightarrow{\text{periodicity}} d = m^2$, m integer

II $r^2R'' + rR' - m^2 R = 0, \qquad 0 \leq r < 1$

equidimensional

Try $R = r^p \Rightarrow$ find p from ODE $\quad p(p-1) + p - m^2 = 0$

$$\begin{cases} p = \pm m,\ m \neq 0 & \rightarrow R(r) = r^{\pm m} \\ p = 0,\ m = 0 \text{ (double)} & \rightarrow R(r) = 1,\ \ln r \end{cases}$$

suppose $m \geq 0$, some of the solutions blow up at zero

this is not a problem for the $\nabla^2 u = 0$

∇^2 can be defined even if a function blows up at 0

$\Rightarrow$ we have a new "physical" assumption $u(0,\theta)$ finite $\circledast$ "physical" assumption

$R(0)$ finite $\Rightarrow R(r) = r^m, 1$

Step 3 Apply BC:

Condition at $r=1$: $\quad u(1,\theta) = f(\theta)$

$u(r,\theta) = r^m e^{\pm im\theta}$

How to satisfy the BC? Take a linear supperposition of all possible u's:

$$u(r,\theta) = \sum_{m=0}^{+\infty} A_m r^m e^{\pm i m\theta} = A_0 + \sum_{m=1}^{\infty}\left[A_m \cos(m\theta) + B_m \sin(m\theta)\right] r^m$$

at $r=1$

$$u(r=1,\theta) = A_0 + \sum_{m=1}^{\infty}\left[A_m \cos(m\theta) + B_m \sin(m\theta)\right] = f(\theta) \Rightarrow \left\{\begin{array}{l} A_m, A_0 \\ B_m \end{array}\right.$$

can be found (Fourier series)

Example of Nonlinear PDE $\Rightarrow$ laplace equ.

Potential functions : help converts systems of PDE's to a linear PDE

<u>Example</u>: Inviscid, incompressible fluid

$\vec{u} = (u^1, u^2, u^3)$, $\vec{r} = (x_1\, x_2\, x_3)$

PDE's
$$\begin{cases} \vec{u}_t + \vec{u}\cdot\nabla\vec{u} = -\underset{\downarrow\,\text{pressure}}{\nabla p} + \vec{F} \rightarrow \text{external force (given)} \\ \nabla\cdot\vec{u} = 0 \\ \nabla\times\vec{u} = 0 \end{cases}$$

$$\nabla\vec{u} = \begin{pmatrix} \frac{\partial u^1}{\partial x_1} & \frac{\partial u^2}{\partial x_1} & \frac{\partial u^3}{\partial x_1} \\ \frac{\partial u^1}{\partial x_2} & \ddots & \\ \frac{\partial u^1}{\partial x_3} & & \ddots \end{pmatrix} \qquad \vec{u}\cdot\nabla\vec{u} = (u^1\, u^2\, u^3)\begin{pmatrix} \ddots & & \\ & \ddots & \\ & & \ddots \end{pmatrix}$$

<u>Recall</u>: In ~~electrics~~ electrostatics, the electric field $\vec{E}$ satisfies $\begin{cases} \nabla\cdot\vec{E} = 0 \\ \nabla\times\vec{E} = 0 \end{cases}$ in source free region.

E satisfies 2 equations but it is not necesscrily zero.

$\begin{cases} \nabla\cdot\vec{u} = 0 \\ \nabla\times\vec{u} = 0 \end{cases}$ we solve this by introducing a potential function

$\vec{u} = -\nabla\phi$ ($\nabla\times\vec{u} = 0$ is satisfied)

from $\nabla\cdot\vec{u} = 0 \Rightarrow$ $\boxed{\nabla^2\phi = 0}$ (additive const often present)

$$-\frac{\partial}{\partial t}\nabla\phi + \nabla\phi\cdot\nabla\nabla\phi = -\nabla p + \underset{-\nabla f}{\vec{F}}$$

(suppose F is conservative $\vec{F} = -\nabla f$)

$$\Rightarrow \nabla p = \nabla\phi_t - \nabla\phi\cdot\nabla\nabla\phi - \nabla f = \nabla\left(\phi_t + \tfrac{1}{2}(\nabla\phi)^2 - f\right) \Rightarrow \boxed{p = \phi_t - \tfrac{1}{2}(\nabla\phi)^2 - f + K(t)}$$

$K(t)$: can be found fro BC