February 17, 2004 Lecture 4

Rev. Session: Friday, 4-5pm

Opt. Reading: Whitham 2.1 - 2.6, 2.12, 3.1

" 5.1 - 5.4

Debnath 4.2 - 4.3, 4.5

Review General 1st-order PDE: $H(x, y, u, p, q) = 0$ $p = u_x$, $q = u_y$

Charpit's eqns: $\frac{dx}{H_p} = \frac{dy}{H_q} = \frac{du}{pH_p + qH_q} = -\frac{dp}{H_x + H_u p} = -\frac{dq}{H_y + H_u q}$

CHAR is a curve in (x, y, u, p, q) space

Example: Eikonal equation (from greek "eikon" = "image")

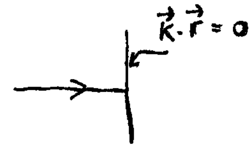
Wave equ: $\nabla^2 \psi = \frac{1}{c^2}\frac{\partial^2 \psi}{\partial t^2}$, $c =$ const $\nabla^2 = \frac{\partial}{\partial x^2} + \frac{\partial}{\partial y^2}$

1D (x): $\frac{\partial^2 \psi}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 \psi}{\partial t^2}$

particular solution: $\psi(x,t) = \overset{\text{const}}{A}\, e^{ikx - i\omega t}$, $\omega, k =$ const, $\frac{\omega}{k} = c$

2D (x, y): $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$, $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\psi = \frac{1}{c^2}\frac{\partial^2 \psi}{\partial t^2}$

particular solution: $\psi(x,y,t) = \underset{\text{const}}{A}\, e^{i\vec{k}\cdot\vec{x} - i\omega t}$ $\vec{k}$ - wave vector



the wave propagates along the wave vector $\vec{k}$

a plane $\perp$ to $\vec{k}$ has the property that $\vec{k}\cdot\vec{r} =$ const

this is called "wavefront" (plain wave, the wf is a plane)

2D Generally, $\psi(x,y,t) = \phi(x,y) \cdot e^{-i\omega t}$ (more complicated than pl. w.)

(for a linear equation, imaginary and real part are both sol.)

satisfies the Helmholtz equ.

$$\nabla^2 \psi = \frac{1}{c^2}\underbrace{\frac{\partial^2 \psi}{\partial t^2}}_{-\omega^2 \psi} \Rightarrow \boxed{\nabla^2 \phi + k^2 \phi = 0}, \quad c = \frac{\omega}{k}$$

Why we do 2nd order PDE? $\nearrow$ can reduce to 1st order PDE

$\searrow$ see the Eikonal Eq.

Seek solutions $\phi(x,y) = \underbrace{B(x,y)}_{\text{real}}\, e^{ik \overbrace{S(x,y)}^{\text{real}}}$ $\quad k = |\vec{k}| = \frac{\omega}{c}$

it is not an assumption: we can always do this, ϕ is a complex function

region of points where $S(x,y) = \text{const}$ is the wave front (general, not necessarily a plane)

Substitute in the second equation:

$$\phi_x = (B_x + ikS_xB)\, e^{ikS}$$

$$+\begin{cases} \phi_{xx} = (B_{xx} + ikS_{xx}B + 2ikS_xB_x - k^2 S_x^2 B)\, e^{ikS} \\ \phi_{yy} = (B_{yy} + ikS_{yy}B + 2ikS_yB_y - k^2 S_y^2 B)\, e^{ikS} \end{cases}$$

HE: $\quad B_{xx} + B_{yy} - k^2 B(S_x^2 + S_y^2 - 1) + 2ik(S_xB_x + S_yB_y) + ikB(S_{xx} + S_{yy}) = 0$

Want PDEs for B, S:

$$\begin{cases} B_{xx} + B_{yy} - \boxed{k^2 B(S_x^2 + S_y^2 - 1)} = 0 \\ 2S_xB_x + 2S_yB_y + B(S_{xx} + S_{yy}) = 0 \end{cases}$$

coupled system of equations

Eikonal approximation: assume that the frequency is high

$k = |\vec{k}| = \frac{\omega}{c}$ $\quad \omega$ high $\Rightarrow$ $k \sim \infty$

and B, S and their derivatives are bounded

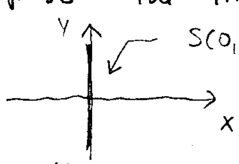
$$\Rightarrow \boxed{S_x^2 + S_y^2 - 1 = 0}$$

first order PDE for the phase

Under the high frequency, S can be solved separatly.

Apply $H(x,y,u,p,q) = 0$ $\quad u \equiv S$ $\quad$ with $H = p^2 + q^2 - 1 = 0$

suppose the initial conditions:



Charpit's eqns: $\quad H_p = 2p, \; H_q = 2q, \; H_x = H_y = H_u = 0$

$$\frac{dx}{2p} = \frac{dy}{2q} = \frac{du}{2p^2 + 2q^2} = -\frac{dp}{0} = -\frac{dq}{0}$$

$\downarrow$ ① p is const $\quad p = k_1$

$\downarrow$ ② q is const $\quad q = k_2$

4equations, sol. true along the CHAR in 5D-space (x, y, S, p, q)

① $p = K_1$

② $q = K_2$

③ $\frac{dx}{2p} = \frac{dy}{2q} \Rightarrow K_2 dx = K_1 dy \Rightarrow K_2 x - K_1 y = K_3 \Rightarrow qx - py = K_3$

④ $\frac{dx}{2p} = \frac{dS}{2p^2 + 2q^2} \Rightarrow \underbrace{(K_1^2 + K_2^2)}_{\text{this's 1 from the E. eq.}} dx = K_1 dS \Rightarrow S = \frac{x}{K_1} + K_4 = \frac{x}{p} + K_4$

these are the CHAR → lines

Apply the initial data: $(x = 0)$

$q|_{s=0} = u_0'(s)$

$p|_{s=0} = [1 - u_0'^2(s)]^{1/2}$

parametrize the data: $\begin{cases} y = s \\ S(0, y) = u_0(s) \end{cases}$

① $p = \sqrt{1 - q_0^2(s)} = K_1 = p_0(s) \Rightarrow K_1 = p_0(s) \Rightarrow p = p_0(s)$

② $q = q_0(s) = K_2 \Rightarrow K_2 = q_0(s) \Rightarrow q = q_0(s)$

③ $q_0 \times 0 - p_0 \times s = K_3 \Rightarrow K_3 = -s p_0(s) \Rightarrow q_0(s) x - p_0(s) y = -s p_0(s)$

④ $u_0(s) = \frac{0}{p_0(s)} + K_4 \Rightarrow K_4 = u_0(s) \Rightarrow S = \frac{x}{p_0(s)} + u_0(s)$

the integration constant has been expressed in terms of the initial data

What to do next?

1) x, y, S, p, q, s 6 variables, 4 equations ⇒ 2 indep. variables → $S(x, y)$ (eliminate s, p, q ... as done last time)

2) study how the solution S depend on s

$x, y, S, \cancel{p}, \cancel{q}, s, \cancel{p_0}, \cancel{q_0}$ parameters and variables appearing in the solution

we want to eliminate p, q, p_0, q_0

⇒ we will find how the data influences the solution S

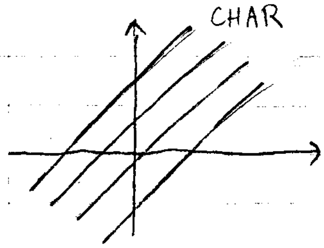
③ $\Rightarrow q_0 x = p_0 (y - s) \Rightarrow q_0^2 x^2 = p_0^2 (y - s)^2$

$(1 - p_0^2) x^2 = p_0^2 (y - s)^2 \Rightarrow x^2 = p_0^2 [x^2 + (y - s)^2]$

④ $\Rightarrow S - u_0 = \frac{x}{p_0} \Rightarrow x^2 = p_0^2 (S - u_0)^2$

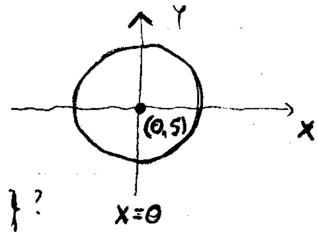
$\Rightarrow 1 = \frac{x^2 + (y - s)^2}{(S - u_0)^2} \Rightarrow \boxed{(S - u_0(s))^2 = x^2 + (y - s)^2}$

how the solution at point (x, y) depends on the Initial data



let $S = \text{const}$

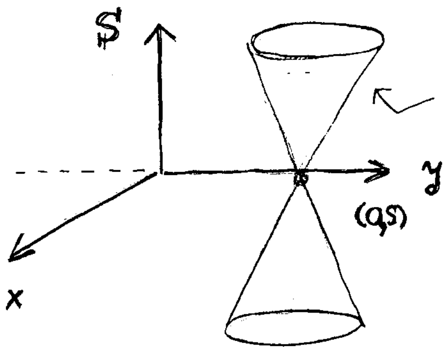
the the equation tells us we have a circle



when S changes the radius changes

What this makes in the $\{x, y, S\}$?

on this plane $S' = u_0$



$$(S-u_0)^2 = x^2 + (y-s)^2$$

Monge cone
the ensemble of points that are affected by the data point $(0,S)$

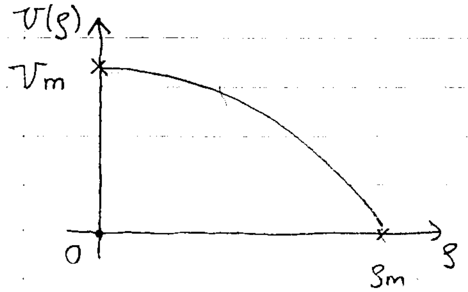
by the point $(0,S)$ the CHAR is a ray : this is how the solution propagates.
by how many ways a ray can affect the solution : turn it around , it gives the Monge cone

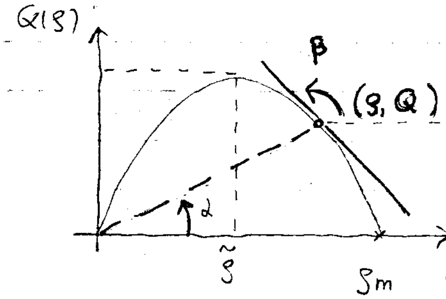
Traffic flow

Recall conservation law $\underbrace{\frac{\partial \rho}{\partial t}}_{\text{density}} + \underbrace{\frac{\partial q}{\partial x}}_{\text{flux}} = 0$

these two quantities are related by the conservation of mass.
In traffic flow : $\begin{cases} \rho = \text{density of cars} & [\text{cars / mile}] \\ q = \text{flux of cars} & [\text{cars / hr}] \end{cases}$
what is conserved is the number of cars.
Statistical car speed $V = \frac{q}{\rho}$ (given)
we assume $V = V(\rho)$
then we can calculate $Q(\rho) = q(\rho) = \rho V(\rho)$

here: $Q(\rho)$ and $V(\rho)$ are average , not instantaneous values



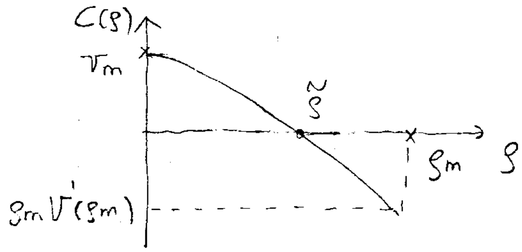


ρ_m ← maximum value of ρ

$tg\,\alpha = \dfrac{Q}{\rho} = V$

since V goes down, the slope has to go down.

$c(\rho) = Q'(\rho)$: propagation speed
i.e. this is the slope $tg\,\beta$



$C(\rho)$

$\Rightarrow$ $c'(\rho) < 0$ decreases
as the density increases, the drivers tend to decrease their speed

PDE for ρ : $\dfrac{\partial \rho}{\partial t} + \underbrace{Q'(\rho)}_{C(\rho)} \dfrac{\partial \rho}{\partial x} = 0$ $\qquad c'(\rho) < 0$

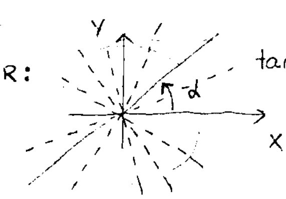
February 18, 2004 Lecture 5

Review Session: FRI 4-5pm
Pick up: Handout 4

<u>Review Lecture 4</u> :

Eikonal eqn. $\begin{cases} S_x^2 + S_y^2 - 1 = 0 \\ S(0,y) = u_0(y), \; y = s \end{cases}$ $\qquad (\, q_0(s) = u_0'(s), \; q_0^2 + p_0^2 = 1 \,)$

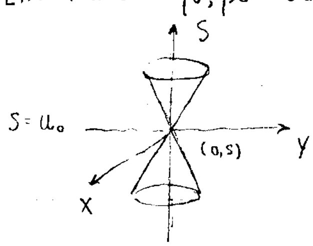
CHAR:



$\tan\alpha = \dfrac{q_0(s)}{p_0(s)}$

$q_0(s)\,x - p_0(s)\,y = K_3$
(for each point $(0,s)$ a line passes through the slope is done by q_0/p_0)

Eliminate q_0, p_0 but keep s fixed $\Rightarrow$ consider all possible slope of CHAR



Monge cone

(all points lying on the disk are affected by the data $(0,s)$, all r)
(all ways the CHAR can affect the solution by the data at point $(0,s)$ → 3D cone)