April 21, 2004 Lecture 21

Pick up: Solution to Hmwk 4 (if you have not)

Start on Hmwk 5!

Rev. Session: FRI, 4:15 pm

**Review** Linear Translation invariant PDE:
(uniform PDE)

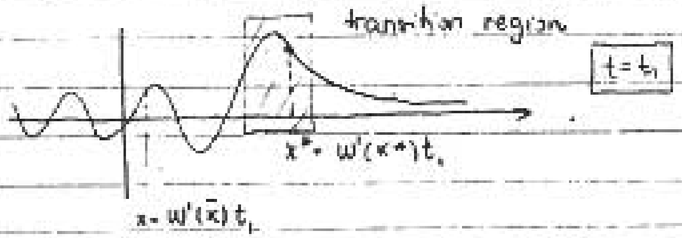
$$\underset{\text{wave}}{u = e^{ikx - i\omega t}} \Rightarrow \omega = W(k) \quad \text{DISPERSION RELATION}$$

FT in x $\left.\begin{matrix} x \to \infty \\ t \to \infty \end{matrix}\right\} \frac{x}{t} = O(1)$ : $\bar{k}: W'(\bar{k}) = \frac{x}{t} \Rightarrow \bar{k} = \bar{k}(x,t)$

↑ group velocity at $k = \bar{k}$

$$I(x,t) \sim \underbrace{\frac{F(\bar{k})}{\sqrt{2\pi|tW'(\bar{k})|}}\, e^{-i\frac{\pi}{4}\operatorname{sgn}(tW''(\bar{k}))}}_{A(x,t)} \underbrace{e^{i\bar{k}x - iW(\bar{k})t}}_{e^{i\theta(x,t)}}$$

Formula breaks down when $\bar{k} \to k^*$, $k^*: W''(k^*) = 0$



Inside transition region

$$I(x,t) \sim \frac{F(k^*)}{(3\gamma t)^{1/3}} \underbrace{\mathrm{Ai}\left(\frac{x - \overset{c_0 = W'(k^*)}{c_0} t}{(3\gamma t)^{1/3}}\right)}_{\text{Airy function}}$$

$\gamma = -\frac{1}{3!}W'''(k^*)$

$$\mathrm{Ai}(z) = \frac{1}{\pi}\sqrt{\frac{z}{3}}\, K_{1/3}\left(\frac{2}{3}z^{3/2}\right)$$

Hankel. Modified Bessel function

Last time we proved:

$$\boxed{\begin{array}{l} \bar{k} = \theta_x \\ W(\bar{k}) = \theta_t \end{array}} \begin{array}{l} \equiv k \quad : \text{local wave number} \\ \equiv \omega \quad : \text{local frequency} \end{array}$$

↳ proved last time

Also: $\left|\frac{(\bar{k})_x}{\bar{k}}\right|$ : "small" means $\ll$ ("characteristic length of the system")$^{-1}$ $\equiv l_c^{-1}$

and any function of this is "small"

$\left|\frac{(\bar{k})_t}{\bar{k}}\right|$ : "small" means $\ll$ ("characteristic time of system")$^{-1}$ $\equiv \tau_c^{-1}$

and any function of this is "small"

This means that the wave of the form:

$$I(x,t) \sim A(x,t)\, e^{i\theta(x,t)}$$

has:

$$\left|\frac{A_x}{A}\right| : \text{"small"}, \quad \left|\frac{A_t}{A}\right| : \text{"small"} \quad \text{etc.}$$

because $\bar{k}$ has this property and enters in the definition of $A, \theta$.
($A, \theta$ are defined as a function of $x,t$ only through $\bar{k}(x,t)$ which is slowly varying)

Such waves are called SLOWLY VARYING $\underbrace{\text{NON UNIFORM}}_{\text{amplitude is not a constant}}$ WAVES

Generalize to Non-uniform PDE's:

**Assume** solution $u(x,t) \sim e^{i\theta(x,t)} \sum_{n=0}^{N} A_n(x,t)$

have to be found

(we have to know that the PDE has a wave-like solution)

**Assumptions:**

(i) $A_0$, $\theta_x$, $\theta_t$ and maybe other coefficients of the PDE are $O(1)$ : "NOT small"

(if you try a wave, the local wavenumber and frequency are fixed (not slowly varying...))

(ii) Increasing the order of derivative in $(x,t)$ by 1, also increases the order of approximation by 1

e.g.

$A_{0,x} = O(\text{"}\varepsilon\text{"})$ , $\theta_{x,x} = O(\text{"}\varepsilon\text{"})$

$A_{0,t} = O(\text{"}\varepsilon\text{"})$ $\theta_{t,t} = O(\text{"}\varepsilon\text{"})$

(anything you differentiate in x, t once gives you an extra ε)

(iii) $\left|\frac{A_n}{A_0}\right| = O(\text{"}\varepsilon^n\text{"})$

(if you increase the order of approximation by 1, you add an extra ε)

Physical foundation for these assumptions: we are trying to find a slowly varying solution.

(we need this ε to describe systematically a sl. var. sol.)

<u>Example</u>: Non-uniform Non-linear Klein-Gordon equation

$$u_{tt} - [\alpha(x,t)^2 u_x]_x + \beta(x,t)^2 u = 0 \quad \begin{cases} -\infty < x < \infty \\ t > 0 \end{cases}$$

Assume $u \sim e^{i\theta(x,t)} \sum_{n=0}^{N} A_n(x,t)$ : "slowly varying"

→ We cannot use Fourier transform because $\alpha(x,t)$, $\beta(x,t)$ depend on x!

Assumptions:

① $A_0$, $\theta_x$, $\theta_t$ , $\alpha$, $\beta$ : $O(1)$

② $A_{0,x}$, $A_{0,t}$, ... : $O(\text{"}\varepsilon\text{"})$

$\alpha_{,x}$ ; $\beta_x$ , .... : $O(\text{"}\varepsilon\text{"})$ etc.

③ $\left|\frac{A_1}{A_0}\right| = O(\text{"}\varepsilon\text{"})$

$A_{1,x} = O(\text{"}c^2\text{"})$

Start $u(x,t) \sim e^{i\theta(x,t)}(A_0 + A_1)$

$$u_x \sim \left[\overset{\varepsilon}{A_{0,x}} + \overset{\varepsilon^2}{A_{1,x}} + i\theta_x(\overset{1}{A_0} + \overset{\varepsilon}{A_1})\right] e^{i\theta}$$

$$u_{xx} \sim \left[\overset{\varepsilon^2}{A_{0,xx}} + \overset{\varepsilon^3}{A_{1,xx}} + 2i\theta_x(\overset{\varepsilon}{A_{0,x}} + \overset{\varepsilon^2}{A_{1,x}}) + \overset{\varepsilon}{i\theta_{xx}}(\overset{1}{A_0} + \overset{\varepsilon}{A_1}) - \theta_x^2(\overset{1}{A_0} + \overset{\varepsilon}{A_1})\right] e^{i\theta}$$

$$u_{tt} \sim \left[A_{0,tt} + A_{1,tt} + i\theta_{tt}(A_0+A_1) + 2i\theta_t(A_{0,t}+A_{1,t}) - \theta_t^2(A_0+A_1)\right]e^{i\theta}$$

$$(\alpha^2 u_x)_x \sim \Big\{2\left[\alpha\alpha_x A_{0,x} + \alpha\alpha_x A_{1,x} + i\alpha\alpha_x\theta_x(A_0+A_1)\right]$$
$$+\alpha^2\left[A_{0,xx} + A_{1,xx} + i\theta_{xx}(A_0+A_1) + 2i\theta_x(A_{0,x}+A_{1,x}) - \theta_x^2(A_0+A_1)\right]\Big\}e^{i\theta}$$

Substitution in the PDE:

$$\left[A_{0,tt} + A_{1,tt} + i\theta_{tt}(A_0+A_1) + 2i\theta_t(A_{0,t}+A_{1,t}) - \theta_t^2(\underset{\sim}{A_0}+A_1)\right]$$

$$-2\alpha\alpha_x\left[A_{0,x} + A_{1,x} + i\theta_x(A_0+A_1)\right]$$

$$-\alpha^2\left[A_{0,xx} + A_{1,xx} + i\theta_{xx}(A_0+A_1) + 2i\theta_x(A_{0,x}+A_{1,x}) - \theta_x^2(\underset{\sim}{A_0}+A_1)\right]$$

$$+\beta^2(A_0+A_1) = 0$$

Collect the terms with equal powers of $\varepsilon$:

Terms of order $\varepsilon^0$:

$$-\theta_t^2 A_0 + \theta_x^2\alpha^2 A_0 + \beta^2 A_0 = 0$$

$$\left[-\theta_t^2 + \alpha^2\theta_x^2 + \beta^2\right]A_0 = 0 \Rightarrow \boxed{-\theta_t^2 + \alpha^2\theta_x^2 + \beta^2 = 0}$$

(I) $\boxed{\alpha^2\theta_x^2 + \beta^2 = \theta_t^2}$ "Dispersion relation" but local
($\omega = -\theta_t$ , $\kappa = \theta_x$)

1st order PDE

(any 1st order PDE is in principle solvable with the method of CHAR
**PDE ⇒ ODE**)

Terms of order $\varepsilon$:

$$\cancelto{0}{(\alpha^2\theta_x + \beta^2 - \theta_t^2)}A_1 + i\left(\theta_{tt}A_0 + 2\theta_t A_{0,t} - 2\alpha\alpha_x\theta_x A_0 - \alpha^2\theta_{xx}A_0 - 2\alpha^2\theta_x A_{0,x}\right) = 0$$

(II) $\boxed{(-2\theta_t)A_{0,t} + 2\alpha^2\theta_x A_{0,x} = \left(\theta_{tt} - 2\alpha\alpha_x\theta_x - \alpha^2\theta_{xx}\right)A_0}$

1st order PDE $\overset{CHAR}{\Rightarrow}$ solvable ⇒ ODE's
(coupled system of ODE's)

Try to solve eq. ⓶ by CHAR:

$$\frac{dt}{-2\theta_t} = \frac{dx}{2\alpha^2\theta_x} = \frac{dA_0}{(\theta_{tt} - 2\alpha\alpha_x\theta_x - \alpha^2\theta_{xx})A_0}$$

slope of CHAR:

$$\frac{dx}{dt} = \frac{\alpha^2\theta_x}{(-\theta_t)}$$

from eq. ⓵ $\alpha^2\theta_x^2 + \beta^2 = \theta_t^2$ by def $\omega = -\theta_t$, $\kappa = \theta_x$

$$\alpha^2\kappa^2 + \beta^2 = \omega^2$$

preserve the order of the dispersion relation:

$(\kappa \to \kappa + d\kappa, \quad \omega \to \omega + d\omega)$ $\alpha, \beta$ not varying, local

$$\alpha^2 2\kappa d\kappa = 2\omega d\omega \Rightarrow \frac{d\omega}{d\kappa} = \frac{\alpha^2\kappa}{\omega}$$

Thus, slope of CHAR is:

$$\frac{dx}{dt} = \frac{\alpha^2\theta_x}{(-\theta_t)} = \frac{\alpha^2\kappa}{\omega} = \frac{d\omega}{d\kappa} = \omega'(\kappa)$$

↳ group velocity

"values of $A_0$ propagate on curves of slope the group velocity"

We want to find connections between hyperbolic equations and dispersive systems:

(i) Recall:

A conservation law is a PDE of the form:

$$\frac{\partial \rho}{\partial t} + \frac{\partial q}{\partial x} = 0$$

density ↙ ↳ flux

Dispersive systems:

local wave number and local frequency satisfy

local : $\begin{cases} \omega \equiv -\theta_t \Rightarrow \omega_x = -\theta_{xt} \\ \kappa \equiv \theta_x \Rightarrow \kappa_t = \theta_{tx} \end{cases}$

$$\boxed{\frac{\partial \kappa}{\partial t} + \frac{\partial \omega}{\partial x} = 0}$$

↓

conservation law

wave number: density of wave crests

frequency : flux of wave crests

what is conserved in a dispersive system, is the number of wave crests.

(ii) Recall: Constitutive relation $q = Q(\rho)$ => PDE for $\rho$

Dispersive systems:

the constitutive relation is the dispersion relation $\omega = W(k)$

$$\frac{\partial k}{\partial t} + \frac{\partial \omega}{\partial x} = 0 \quad \Rightarrow \quad \boxed{\frac{\partial k}{\partial t} + W'(k)\frac{\partial k}{\partial x} = 0}$$

The $c_g$ is the speed of propagation of constant values of $k$.

-"- is the slope of the CHAR.

The speed of propagation of const $k = k_0$ is equal to $W'(k_0)$

Shocks: In traffic flow (based on density) shocks are sign of a pathology. Here, multiple valued solutions are allowed (not excluded a priori).