

Problems 12 and 13 serve as exercises in finding particular solutions of PDEs by techniques such as separation of variables or guess of the *form* of the solution. The description of the references in parentheses can be found in the Bibliography for 18.306.

12. In two space dimensions (x, y) , the time-dependent, linear Schrödinger equation (LSE), which governs the wave motion of a non-relativistic particle, is $i\Psi_t = -(\Psi_{xx} + \Psi_{yy}) + V(x, y)\Psi$, where $i^2 = -1$, $\Psi = \Psi(x, y, t)$ is the particle wave function to be found, and $V(x, y)$ is a given, external potential.
- (a) Can you solve LSE by separation of variables in x, y and t if $V(x, y) = V_1(x) + V_2(y)$ where V_1 and V_2 are *arbitrary*? Explain.
- (b) Now consider $V = xy$. Identify new variables x_1 and x_2 as suitable transformations of x and y so that the LSE is solved by separation of variables in x_1, x_2 and t . Show your steps.
- (c) Answer part (a) with V replaced by $V(x, y) = V_1(x) V_2(y)$ where V_1 and V_2 are reasonably *arbitrary* functions of one variable.
13. (Evans, Prob. 4.7.6, p. 234.) The flow velocity $v(z, t)$ of a diffusing dye satisfies the PDE

$$v_t - v_{zz} + \beta(zv)_z + (\dot{\sigma}/\sigma)v = 0, \quad t > 0, \quad -\infty < z < +\infty,$$

where $\beta = \beta(t)$ and $\sigma = \sigma(t)$ are given functions of time, and the dot on top of σ denotes its time derivative. It is also given that $v \rightarrow 0$ as $z \rightarrow \pm\infty$ and $v(z, 0) = \delta(z)/\sigma(0)$ where $\delta(z)$ is the delta function. An experimentalist observes that v has the form of a Gaussian with (unknown) variable width $\gamma(t)$,

$$v(z, t) = \sigma(t)^{-1} [\pi\gamma(t)]^{-1/2} e^{-z^2/\gamma(t)}.$$

Find the ODE satisfied by $\gamma(t)$. What is the initial condition for this ODE that $\gamma(t)$ satisfies? Find $\gamma(t)$ for $t > 0$ by solving the initial-value problem explicitly.

14. (a) Find explicitly the Green function for the modified Helmholtz operator $(\nabla^2 - k^2)$ in the three-dimensional (3D) space for $k > 0$, i.e., determine the $G(\mathbf{r}, \mathbf{r}')$ that satisfies $(\nabla^2 - k^2)G(\mathbf{r}, \mathbf{r}') = \delta(\mathbf{r} - \mathbf{r}')$ and is bounded at infinity; $\mathbf{r} = (x, y, z)$ is the position vector in the 3D infinite space.
- (b) Find explicitly the Green function $G(\mathbf{r}; \mathbf{r}')$ for outgoing waves that is specified by the solution $w = G(\mathbf{r}; \mathbf{r}')e^{i\omega t}$ of the PDE

$$c^{-2}w_{tt} + \nabla^2 w + \alpha^2 \nabla^2 (\nabla^2 w) = \delta(\mathbf{r} - \mathbf{r}')e^{i\omega t},$$

where $\alpha > 0, c > 0$. **Hint:** $\nabla^2 w = r^{-1} \frac{d^2}{dr^2}(rw)$. G should NOT have any terms with exponential growth or decay for large r . Outgoing waves have phase fronts that move away from the origin as t increases. (This kind of boundary condition is called *Sommerfeld radiation condition* but you don't need to know this fact in order to solve the problem).