18.306: Advanced PDE with Applications   Friday, 04/23/04

Handout 8: Summary of Lectures 20, 21. on Wave Dispersion.

LECTURE 20:

A linear, translationally invariant ("uniform") PDE may have a dispersion relation $\omega = W(k)$, which results from trying a wave solution $u = e^{ikx - i\omega t}$, or making the replacements

$$\frac{\partial}{\partial t} \equiv -i\omega \quad \text{and} \quad \frac{\partial}{\partial x} \equiv ik \qquad \text{in the PDE.}$$

By linear superposition, the complete solution to the PDE is then given in terms of the Fourier integral

$$I(x,t) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} \, e^{ikx - iW(k)t} \, F(k)$$

F.T. amplitude from data

$$\simeq \frac{F(\bar{k})}{\sqrt{2\pi \cdot |t \cdot W''(\bar{k})|}} \cdot e^{i\bar{k}\cdot x - iW(\bar{k})\cdot t} \cdot e^{-i\frac{\pi}{4}\operatorname{sgn}(t\cdot W''(\bar{k}))} \qquad (1)$$

DISPERSIVE WAVE

where $x \to \infty$, $t \to \infty$ with $\frac{x}{t} = O(1)$, and $\bar{k}$ is defined so that

$$W'(\bar{k}) = \frac{x}{t}. \qquad \therefore \quad \bar{k} = \bar{k}(x,t) : \text{local wave number.}$$

In order to derive the last formula, we assumed that "$t \to \infty$" with $\frac{x}{t}$: "fixed", i.e., not growing in time, and applied the stationary-phase method. The condition "$t \to \infty$" really means $t \gg \tau_c$: characteristic time of the system, while the condition "$x \to \infty$" means $x >> \ell_c$: characteristic length of the system. Both ℓ_c and τ_c can be found via dimensional arguments from (dimensional) parameters in the dispersion relation $\omega = W(k)$.

Example: Find ℓ_c and τ_c for a system described by the Klein-Gordon equation, $u_{tt} - \alpha^2 u_{xx} + \beta^2 u = 0$; $\alpha, \beta > 0$.

Answer: The dispersion relation reads $\omega^2 \equiv W(k)^2 = \alpha^2 k^2 + \beta^2$, where α has dimension $\frac{\text{Length}}{\text{Time}}$ and β has dimension $\frac{1}{\text{Time}}$, because the frequency ω has dimension $\frac{1}{\text{Time}}$ and the wave number k has dimension $\frac{1}{\text{Length}}$. The quantities ℓ_c and τ_c are thus formed dimensionally from α and β or their combinations:

$$\tau_c = \frac{1}{\beta} \quad (\text{units of Time}); \qquad \ell_c = \frac{\alpha}{\beta} \quad (\text{units of Length})$$

– End of Example –

The meaning of $W'(\bar{k})$ from Eq. (1) is the following. An observer who wants to see the <u>same</u> local wave number $\bar{k}$ and local frequency $W(\bar{k})$ <u>must</u> move with speed $W'(\bar{k})$.

The quantity $W'(k)$ is called the <u>group velocity</u>.

The meaning of the group velocity becomes more transparent by considering the "energy" $Q(t)$ of a wave packet in the region from $x = x_1$ to $x = x_2$:

$$Q(t) \underset{\text{proportional to}}{\propto} \int_{x_1}^{x_2} dx \; |I(x,t)|^2 \overset{\text{Eq. (1)}}{\simeq} \int_{x_1}^{x_2} dx \; \frac{|F(\bar{k}(x,t))|^2}{2\pi \cdot |t \; W''(\bar{k}(x,t))|} \; . \quad (x_1 < x_2).$$

With the change of variable $x = W'(\bar{k}) \cdot t$ so that $x \Rightarrow \bar{k}$: new integration variable, $Q(t)$ is

$$Q(t) \propto \int_{\bar{k}_1}^{\bar{k}_2} d\bar{k} \; W''(\bar{k}) \cdot t \; \frac{|F(\bar{k})|^2}{2\pi \cdot |t \cdot W''(\bar{k})|} = \int_{\bar{k}_1}^{\bar{k}_2} \frac{d\bar{k}}{2\pi} \cdot |F(\bar{k})|^2 , \quad \text{assuming } W''(\bar{k}) > 0 \text{ for } \bar{k}_1 < \bar{k} < \bar{k}_2 ;$$

note that $\bar{k}_1, \bar{k}_2$: $x_1 = W'(\bar{k}_1) \cdot t$ and $x_2 = W'(\bar{k}_2) \cdot t$.

Hence, the energy $Q(t)$ contained in the wave packet between the space points x_1 and x_2 remains <u>constant</u> only if $\bar{k}_1$ and $\bar{k}_2$ are constants (fixed).

But $\bar{k}_1$: const. and $\bar{k}_2$: const. means that $x_1 = x_1(t)$ & $x_2 = x_2(t)$ move with the group velocity.

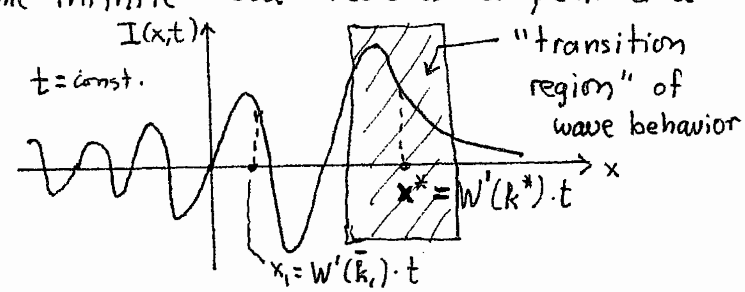
– End of Discussion on group velocity –

There is an exceptional case in which formula (1) breaks down:

<u>if</u> there is a k^* such that $W''(k^*) = 0$, then

as $W''(\bar{k}) \to 0$ when $\bar{k} \to k^*$, RHS of (1) $\to \infty$.

The RHS of (1) then describes a wave with increasing amplitude; in reality, the amplitude does not become infinite but reaches a peak and then decays exponentially in x:



This means that, for $x > x^*$ (and t: fixed), there is <u>no</u> <u>real</u> k that satisfies

$$W'(k) = \frac{x}{t}.$$

<u>For example</u>, consider $W(k) = ak - b \cdot k^3$, $a, b > 0$. Hence, $W''(k) = 0$ at $k = 0$. It follows that $W'(k) = a - 3bk^2 \leq a$, i.e., the slope $W'(k)$ of $\omega = W(k)$ <u>cannot</u> exceed the value a. So, there is no real k that satisfies $W'(k) = \frac{x}{t}$ if $\frac{x}{t} > a$. — End of example —

In general, the line $\frac{x^*}{t} = W'(k^*)$ is called CAUSTIC.

To describe mathematically the wave inside the transition region we study the original Fourier integral from p. 1, $I(x,t)$.

For the sake of simplicity, we assume that (usually true for water waves)

$F(k)$: even function of k,

$W(k)$: odd function of k. Then,

$$I(x,t) = \int_0^\infty \frac{dk}{\pi} \, F(k) \cdot \cos[kx - W(k) \cdot t]. \qquad (2)$$

For (x,t) near (x^*,t) we approximate $F(k)$ and $W(k)$ near $k = k^*$:

$$F(k) \simeq F(k^*), \qquad W(k) \sim W(k^*) + (k-k^*) \, W'(k^*) + \frac{1}{2}(k-k^*)^2 \cancel{W''(k^*)}^{\,\to 0}$$
$$+ \frac{1}{3!}(k-k^*)^3 \, W'''(k^*).$$

Because $W''(k)$ is an <u>odd</u> function of k, $W''(0) = 0$. So, <u>take</u> <u>$k^* = 0$</u>.
Hence, $W(k^*) = W(0) = 0$ because $W(k)$ is odd.

Equation (2) then becomes

$$I(x,t) \simeq F(k^*) \cdot \int_0^\infty \frac{dk}{\pi} \cos\left[k\cdot(x - c_0 t) + \gamma \cdot t\, k^3 \right],$$

where $c_0 \equiv W'(0)$: group velocity at $k^* = 0$ and $\gamma \equiv -\frac{1}{3!} W'''(0)$.

The last integral can be expressed in terms of a tabulated, special function, the Airy function $Ai(z)$ defined as

$$Ai(z) \equiv \int_0^\infty \frac{dk}{\pi} \cdot \cos\left(kz + \frac{1}{3} k^3\right).$$

It follows that

$$I(x,t) \simeq F(k^*) \cdot \frac{1}{(3\gamma t)^{1/3}} \cdot Ai\left[\frac{x - c_0 t}{(3\gamma t)^{1/3}} \right]$$

gives wave behavior inside transition region, from oscillatory to exponentially decaying behavior

Digression: The Airy function $Ai(z)$ is related to the modified Bessel function $K_{1/3}(\zeta)$ with $\zeta = \frac{2}{3} z^{3/2}$:

$$Ai(z) = \frac{1}{\pi} \sqrt{\frac{z}{3}}\; K_{1/3}\left(\frac{2}{3} z^{3/2}\right) \quad ; \quad z = \frac{x - c_0 t}{(3\gamma t)^{1/3}} \text{ from above.}$$

Recall that $y = K_\nu(\zeta)$ satisfies the modified-Bessel ODE

$$\zeta^2 \cdot y''(\zeta) + \zeta \cdot y'(\zeta) - (\zeta^2 + \nu^2)\, y(\zeta) = 0 \quad \text{with} \quad y(\zeta) \to 0 \text{ as } \zeta \to +\infty.$$

— End of Digression —

$Ai(z)$ has the asymptotic behavior

$$Ai(z) \sim \begin{cases} \dfrac{1}{2\sqrt{\pi}} \cdot z^{-1/4} \cdot e^{-\frac{2}{3} z^{3/2}}, & z \to +\infty \text{ (large } \underline{\text{positive}}\ z) \\ \dfrac{1}{\sqrt{\pi}}\, |z|^{-1/4} \cdot \sin\left(\frac{2}{3} |z|^{3/2} + \frac{\pi}{4}\right), & z \to -\infty \text{ (large } \underline{\text{negative}}\ z). \end{cases}$$

The last formula describes explicitly the wave behavior in the regime of oscillations ($x < c_0 t = x^*$, $z \to -\infty$) and in the regime of exponential decay ($x > c_0 t = x^*$, $z \to +\infty$) ; assume $\gamma > 0$.
More precisely, the wave solution oscillates for $c_0 t - x \gg (3\gamma t)^{1/3}$ and decays exponentially for $x - c_0 t \gg (3\gamma t)^{1/3}$. Thus, the "width" of the transition region is understood to be $O[(\gamma t)^{1/3}]$.

This discussion is quite general, assuming $F(k)$: even and $W(k)$: odd. Similar conclusions hold for other "classes" of $F(k)$ and $W(k)$, assuming that a k^* exists such that $W''(k^*) = 0$.

Meaning of k^* and wave transition: If $W(k)$ is sufficiently smooth in k, the group velocity, $W'(k)$, attains a local extremum at $k = k^*$. This means that $|W'(k)|$ obtains a value which is a maximum over some range of frequencies. Hence, an observer who moves faster than this maximum value cannot observe any local wavenumber $\bar{k}$ and, instead, sees the wave decay exponentially.

— End of Discussion of Wave Transition —

Back to Eq. (1) of p. 1: When there are more than one points $\bar{k}$ such that $W'(\bar{k}) = \frac{x}{t}$, say points $\bar{k}_j$, $j = 1, \dots, N$, then the corresponding formula obtained by the stationary-phase method reads:

$$I(x,t) \simeq \sum_{\{\bar{k}_j\}} \frac{F(\bar{k}_i)}{\sqrt{2\pi \cdot |t\, W''(\bar{k}_j)|}} \cdot e^{-i\frac{\pi}{4}\operatorname{sgn}[t \cdot W''(\bar{k}_j)]} \cdot e^{i\bar{k}_j \cdot x - iW(\bar{k}_j)\cdot t}$$

– End of Comment on multiple $\bar{k}_j$ –

The dispersive wave of Eq. (1) is of the form $I(x,t) \simeq A(x,t) \cdot e^{i\theta(x,t)}$, where $\theta(x,t) = \bar{k}\cdot x - W(\bar{k})\cdot t$; $A(x,t)$ is a <u>non-uniform</u> amplitude since it varies in x, t via $\bar{k}$.

We note that :

- $-\theta_t = -\bar{k}_t \cdot x + W'(\bar{k})\cdot t\ \bar{k}_t + W(\bar{k}) = -\bar{k}_t \cdot \underbrace{[x - W'(\bar{k})\cdot t]}_{=0} + W(\bar{k})$: local frequency

Hence, the <u>local frequency</u> of the (non-uniform) wave is

$$\boxed{\omega \equiv -\theta_t .}$$

- Also : $\theta_x = \bar{k}_x \cdot x + \bar{k} - W'(\bar{k})\cdot t\ \bar{k}_x = \bar{k}_x \cdot \underbrace{[x - W'(\bar{k})\cdot t]}_{=0} + \bar{k} = \bar{k}$: local wave #.

Hence, the <u>local wave number</u> is $\boxed{k \equiv \theta_x}$.

Both k and ω are slowly varying in space and time, in the following sense. From $W'(\bar{k}) = \frac{x}{t}$ we get

differentiate in x

$$\frac{\partial}{\partial x}\cdot \; : \; W''(\bar{k})\cdot \bar{k}_x = \frac{1}{t} \Rightarrow \left|\frac{\bar{k}_x}{\bar{k}}\right| = \left|\frac{W'(\bar{k})}{\bar{k}\cdot W''(\bar{k})} \; \frac{1}{x}\right| \; : \; \text{"small" quantity}$$

differentiate in t

$$\frac{\partial}{\partial t}\cdot \; : \; W''(\bar{k})\cdot \bar{k}_t = -\frac{x}{t^2} \Rightarrow \left|\frac{\bar{k}_t}{\bar{k}}\right| = \left|\frac{W'(\bar{k})}{\bar{k}\cdot W''(\bar{k})} \cdot \frac{1}{t}\right| \; : \; \text{"small"}$$

Since $\left|\frac{\bar{k}_x}{\bar{k}}\right|$ and $\left|\frac{\bar{k}_t}{\bar{k}}\right|$ have dimensions of $\frac{1}{\text{Length}}$ and $\frac{1}{\text{Time}}$, respectively, we infer that

$$\left|\frac{\bar{k}_x}{\bar{k}}\right| \ll (\text{char. length of the system})^{-1} = \ell_c^{-1}$$

$$\left|\frac{\bar{k}_t}{\bar{k}}\right| \ll (\text{char. time of the system})^{-1} = \tau_c^{-1}.$$

It follows that
$$\left|\frac{A_x}{A}\right| \ll \ell_c^{-1}, \quad \left|\frac{A_t}{A}\right| \ll \tau_c^{-1}.$$

We call such a function $A(x,t)$ "slowly varying" in space and time. The last set of relations mean that the average length over which $A(x,t)$ varies is <u>much larger</u> than the characteristic length scale of the system, <u>and</u> that the average time over which $A(x,t)$ varies is <u>much larger</u> than the characteristic time scale of the system.

LECTURE 21 :

The concept of a slowly-varying, non-uniform wave can be generalized to PDE that can not be solved by FT, such as the PDE with non-constant coefficients or nonlinear PDE.

Generalization to Non-uniform and Nonlinear PDE :

Try a solution of form

$$u(x,t) \sim e^{i\theta(x,t)} \cdot \underbrace{\sum_{n=0}^{\infty} A_n(x,t)}_{A(x,t)} ,$$

or

$$u(x,t) \sim \begin{matrix}\mathrm{Re} \\ \mathrm{Im}\end{matrix}\left\{ e^{i\theta(x,t)} \cdot \sum_{n=0}^{\infty} A_n(x,t) \right\} ,$$

where the following "rules" (assumptions) apply:

① θ_x, θ_t, A_0, : $O(1)$, "not small" and coeffs. of PDE (unless otherwise stated)

② Increasing derivative by 1 increases order by 1, i.e., if we think of $A_0 = O(1)$, then

$$\left|\frac{A_{0,x}}{A_0}\right| = \text{"}\varepsilon\text{"}, \quad \left|\frac{A_{0,t}}{A_0}\right| = \text{"}\varepsilon\text{"}, \quad \left|\frac{A_{0,xx}}{A_0}\right| = \text{"}\varepsilon^2\text{"}; \quad \left|\frac{A_{0,xt}}{A_0}\right| = \text{"}\varepsilon^2\text{"} \text{ etc}$$

③ Increasing subscript in A_n by 1 increases order by 1, i.e.,

$$\left|\frac{A_n}{A_0}\right| = \text{"}\varepsilon^n\text{"}.$$

Remark : This "ε" can be thought of a the ratio of ℓ_c or τ_c over the length or time, respectively, over which $A(x,t)$ varies appreciably.

<u>Example</u> : Non-uniform Klein-Gordon eqn, $u_{tt} - [\alpha(x,t)^2 u_x]_x + \beta(x,t)^2 u = 0$, where $\alpha(x,t)$, $\beta(x,t)$: known.

Try $$u(x,t) \sim e^{i\theta(x,t)} \cdot \underbrace{A(x,t)}_{= \sum_{n=0}^{\infty} A_n(x,t)}$$

<u>Assumptions</u> : ① $A_0, \theta_x, \theta_t, \alpha, \beta = O(1)$, "not small"

② $A_0, \theta_x, \theta_t, \alpha$: "slowly varying", i.e.,

$$\left|\frac{A_{0,x}}{A_0}\right| = \text{"}\varepsilon\text{"}, \quad \left|\frac{A_{0,t}}{A_0}\right| = \text{"}\varepsilon\text{"}, \quad \left|\frac{\alpha_x}{\alpha}\right| = \text{"}\varepsilon\text{"}, \quad \left|\frac{\alpha_t}{\alpha}\right| = \text{"}\varepsilon\text{"} \quad \text{etc for } \theta_x, \theta_t,$$

which means that the lengths over which these quantities vary are large compared with the characteristic length of the system, and the times over which A_0, θ_x, θ_t and α vary are large compared with the characteristic time scale of the system..

Increasing derivative by 1 increases order in ε by 1

③ $$\left|\frac{A_n}{A_0}\right| = \text{"}\varepsilon^n\text{"};$$

same rule for derivatives of A_n.

Start with
$$u(x,t) \sim e^{i\theta(x,t)} (A_0 + A_1);$$
we seek equations for θ_0 and A_0.

$$u_x \sim [\overset{\text{"}\varepsilon\text{"}}{A_{0,x}} + \overset{\text{"}\varepsilon^2\text{"}}{A_{1,x}} + i\overset{\text{"}1\text{"}}{\theta_x}(\overset{\text{"}1\text{"}}{A_0} + \overset{\text{"}\varepsilon\text{"}}{A_1})] e^{i\theta},$$

$$u_{xx} \sim [\overset{\text{"}\varepsilon^2\text{"}}{A_{0,xx}} + \overset{\text{"}\varepsilon^3\text{"}}{A_{1,xx}} + i\overset{\text{"}\varepsilon\text{"}}{\theta_{xx}}(\overset{\text{"}1\text{"}}{A_0} + \overset{\text{"}\varepsilon\text{"}}{A_1}) + 2i\overset{\text{"}1\text{"}}{\theta_x}(\overset{\text{"}\varepsilon\text{"}}{A_{0,x}} + \overset{\text{"}\varepsilon^2\text{"}}{A_{1,x}}) - \overset{\text{"}1\text{"}}{\theta_x^2}(\overset{\text{"}1\text{"}}{A_0} + \overset{\text{"}\varepsilon\text{"}}{A_1})] e^{i\theta},$$

$$u_t \sim \left[\overset{\text{"ε"}}{A_{0,t}} + \overset{\text{"ε}^2\text{"}}{A_{1,t}} + i\,\overset{\text{"1"}}{\theta_t}\,(\overset{\text{"1"}}{A_0} + \overset{\text{"ε"}}{A_1}) \right] e^{i\theta},$$

$$u_{tt} \sim \left[\overset{\text{"ε}^2\text{"}}{A_{0,tt}} + \overset{\text{"ε}^3\text{"}}{A_{1,tt}} + i\overset{\text{"ε"}}{\theta_{tt}}\,(\overset{\text{"1"}}{A_0} + \overset{\text{"ε"}}{A_1}) + 2i\overset{\text{"1"}}{\theta_t}\,(\overset{\text{"ε"}}{A_{0,t}} + \overset{\text{"ε}^2\text{"}}{A_{1,t}}) - \overset{\text{"1"}}{\theta_t^2}\,(\overset{\text{"1"}}{A_0} + \overset{\text{"ε"}}{A_1}) \right] e^{i\theta},$$

$$(\alpha^2 u_x)_x = 2\alpha\alpha_x u_x + \alpha^2 u_{xx}$$

$$\sim 2\left[\overset{\text{"1"}}{\alpha}\overset{\text{"ε"}}{\alpha_x}\overset{\text{"ε"}}{A_{0,x}} + \overset{\text{"1"}}{\alpha}\overset{\text{"ε"}}{\alpha_x}\overset{\text{"ε}^2\text{"}}{A_{1,x}} + i\overset{\text{"1"}}{\alpha}\overset{\text{"ε"}}{\alpha_x}\overset{\text{"1"}}{\theta_x}(\overset{\text{"1"}}{A_0} + \overset{\text{"ε"}}{A_1}) \right] e^{i\theta}$$

$$+ \overset{\text{"1"}}{\alpha^2}\left[A_{0,xx} + A_{1,xx} + i\theta_{xx}(A_0 + A_1) + 2i\theta_x(A_{0,x} + A_{1,x}) - \theta_x^2(A_0 + A_1) \right] e^{i\theta}.$$

Substitute into the PDE:

$$\left[\overset{\text{"ε}^2\text{"}}{A_{0,tt}} + \overset{\text{"ε}^3\text{"}}{A_{1,tt}} + i\overset{\text{"ε"}}{\theta_{tt}}(\overset{\text{"1"}}{A_0} + \overset{\text{"ε"}}{A_1}) + 2i\overset{\text{"1"}}{\theta_t}(\overset{\text{"ε"}}{A_{0,t}} + \overset{\text{"ε}^2\text{"}}{A_{1,t}}) - \overset{\text{"1"}}{\theta_t^2}(\overset{\text{"1"}}{A_0} + \overset{\text{"ε"}}{A_1}) \right]$$

$$- 2\left[\overset{\text{"1"}}{\alpha}\overset{\text{"ε"}}{\alpha_x}\overset{\text{"ε"}}{A_{0,x}} + \overset{\text{"1"}}{\alpha}\overset{\text{"ε"}}{\alpha_x}\overset{\text{"ε}^2\text{"}}{A_{1,x}} + i\overset{\text{"1"}}{\alpha}\overset{\text{"ε"}}{\alpha_x}\overset{\text{"1"}}{\theta_x}(\overset{\text{"1"}}{A_0} + \overset{\text{"ε"}}{A_1}) \right]$$

$$- \alpha^2\left[\overset{\text{"ε}^2\text{"}}{A_{0,xx}} + \overset{\text{"ε}^3\text{"}}{A_{1,xx}} + i\overset{\text{"ε"}}{\theta_{xx}}(\overset{\text{"1"}}{A_0} + \overset{\text{"ε"}}{A_1}) + 2i\overset{\text{"1"}}{\theta_x}(\overset{\text{"ε"}}{A_{0,x}} + \overset{\text{"ε}^2\text{"}}{A_{1,x}}) - \overset{\text{"1"}}{\theta_x^2}(\overset{\text{"1"}}{A_0} + \overset{\text{"ε"}}{A_1}) \right]$$

$$+ \beta^2(\overset{\text{"1"}}{A_0} + \overset{\text{"ε"}}{A_1}) = 0.$$

Set equal coefficients of same powers of "ε":

usually called "EIKONAL EQN"

$\underline{\varepsilon^0}$: $\quad -\theta_t^2 \cancel{A_0} + \alpha^2\theta_x^2 \cancel{A_0} + \beta^2 \cancel{A_0} = 0 \iff \boxed{\theta_t^2 = \alpha^2\theta_x^2 + \beta^2}$;

this is the <u>local</u>, leading-order dispersion relation where $\omega \equiv -\theta_t$ and $k \equiv \theta_x$; it's a 1st-order PDE for θ that can be solved by method of characteristics : PDE $\Rightarrow$ ODE

$\underline{\varepsilon^1}$:

$$i\left(\theta_{tt}A_0 + 2\theta_t A_{0,t}\right) - \theta_t^2 A_1 - 2i\alpha\alpha_x\theta_x A_0 - i\alpha^2\theta_{xx}A_0 - 2i\alpha^2\theta_x A_{0,x}$$
$$+ \alpha^2\theta_x^2 A_1 + \beta^2 A_1 = 0$$

$$\Leftrightarrow \quad \cancelto{0}{\left(\alpha^2\theta_x^2+\beta^2-\theta_t^2\right)} A_1 + i\left(\theta_{tt}A_0 + 2\theta_t A_{0,t} - 2\alpha\alpha_x\theta_x A_0 - \alpha^2\theta_{xx}A_0 - 2\alpha^2\theta_x A_{0,x}\right) = 0$$

from dispersion relation
(eqn. from balance in ε^0)

$$\Leftrightarrow \quad -2\theta_t A_{0,t} + 2\alpha^2\theta_x A_{0,x} = \theta_{tt}A_0 - 2\alpha\alpha_x\theta_x A_0 - \alpha^2\theta_{xx}A_0$$

or, $$2(-\theta_t)A_{0,t} + 2\alpha^2\theta_x A_{0,x} = \left(\theta_{tt} - 2\alpha\alpha_x\theta_x - \alpha^2\theta_{xx}\right)A_0 \, ,$$

which is a 1st-order PDE for A_0 given θ.

CHAR : $$\frac{dt}{-2\theta_t} = \frac{dx}{2\alpha^2\theta_x} \quad \Leftrightarrow \quad \frac{dx}{dt} = \frac{\alpha^2\theta_x}{(-\theta_t)}$$

From local dispersion relation $(\omega \equiv -\theta_t, \ k \equiv \theta_x)$:

$$\omega^2 = \alpha^2 k^2 + \beta^2 \quad \Leftrightarrow \quad 2\omega\, d\omega = 2\alpha^2 k\, dk \quad \Leftrightarrow \quad \frac{d\omega}{dk} = \frac{\alpha^2 k}{\omega}$$

(leading-order change in ω and k)

$$\text{CHAR}: \ \frac{dx}{dt} = \underbrace{\omega'(k)}_{\text{GROUP VELOCITY}}$$

i.e., the slope of the characteristics for A_0 is the group velocity.

So, we reduced the Non-uniform PDE to 2 1st-order PDE ;
PDE $\Rightarrow$ ODEs !
Our theme is preserved

— End of Example —

The remarks of p. 12 call for some generalizations.
Deeper reasoning on the role of group velocity with non-uniform ω, k:

$$\left.\begin{array}{l} \omega \equiv -\theta_t \\ k \equiv \theta_x \end{array}\right\} \Rightarrow \quad \omega_x = -k_t \iff \frac{\partial k}{\partial t} + \frac{\partial \omega}{\partial x} = 0$$

RECALL: A conservation law is of form $\dfrac{\partial \rho}{\partial t} + \dfrac{\partial q}{\partial x} = 0$,

where ρ: "density" and q: "flux.

Here, k is the "density" of wave crests and ω is the "flux" of wave crests. So, the PDE for ω and k expresses the conservation of wave crests in non-uniform media!

RECALL: To solve a PDE from a conservation law, we often assume $q = Q(\rho)$, constitutive relation between q and ρ;

$$\frac{\partial \rho}{\partial t} + Q'(\rho)\,\frac{\partial \rho}{\partial x} = 0$$

Then, the value $\rho = \rho_0$ "propagates" along a CHAR with speed $Q'(\rho_0)$

Here, k is related to ω via $\omega = W(k)$: dispersion relation.

Hence,

$$\frac{\partial k}{\partial t} + \frac{\partial \omega}{\partial x} = 0 \quad \Rightarrow \quad \frac{\partial k}{\partial t} + W'(k)\,\frac{\partial k}{\partial x} = 0\,;$$

value $k = k_0$ "propagates" along CHAR with speed $W'(k_0) =$ GROUP VELOCITY!

$\therefore$ Intimate relation between dispersive systems and 1st-order (hyperbolic) PDE