April 7, 2004 Lecture 17

Rev. session FRI 4:15 - 5:30 pm

Extra lecture : FRI, April 16 ; hrs TBA

Review. Self-Adjoint eigenvalue problem: $\left\{ \begin{array}{l} Mu = \mu N u \\ + \text{ homogeneous bc's} \end{array} \right.$ M, N : self-adjoint & linear

u in $\{u_m\}$: complete, orthogonal ; μ in $\{\mu_m\}$

Proof of orthogonality :

$$M u_n = \mu_n N u_n \quad | \, u_k \quad \Rightarrow \quad u_k M u_n = \mu_n \, u_k N u_n$$
$$M u_k = \mu_k N u_k \quad | \, u_n \qquad \quad u_n M u_k = \mu_k \, u_n N u_k$$

subtract + integrate over R

$$\underbrace{\int_R d\vec{r} \, (u_k M u_n - u_n M u_k)}_{0 \ (\mu. \text{ self-adjoint})} = \int_R d\vec{r} \, (\mu_n u_k N u_n - \mu_k u_n N u_k)$$
$$= (\mu_n - \mu_k) \int_R d\vec{r} \, u_k N u_n$$

because $\int d\vec{r} \, u_k N u_n = \int d\vec{r} \, u_n N u_k$

$$\Rightarrow \boxed{\int_R d\vec{r} \, u_k N u_n = 0 \quad \text{if} \quad \mu_k \neq \mu_n}$$: orthogonality of u_k & u_n

Green's functions : Help convert PDE's to integral equations
can be used to solve linear as well as non-linear problems;
for non-linear: it helps to get at least a first approximation of the solution

Example: Static (No time-dependent) charges create electrostatic fields $\vec{E}(\vec{r}) \rightarrow \vec{E} = -\nabla \phi$ potential
a charge distribution $\rho(\vec{r})$ creates a potential $\phi(\vec{r})$
(if you know ϕ, you know everything in electrostatics)

$$\phi(\vec{r}) = \iiint d\vec{r}' \, \rho(\vec{r}') \underbrace{\frac{1}{4\pi |\vec{r} - \vec{r}'|}}_{\text{Coulomb law empirical}} \rightarrow \text{potential } \phi^\delta(\vec{r}) \text{ created by an unit point charge at } \vec{r}'$$

physical principle behind: $\boxed{\text{superposition}}$

if I know the response to something infinitesimal, I can find the response to any forcing.

PDE viewpoint:

the potential satisfies the Poisson equation

("source": has to decay sufficiently fast)

$$\begin{cases} \nabla^2 \phi(\vec{r}) = -\rho(\vec{r}) & \text{(Non-homogeneous PDE)} \\ \phi \to 0 \quad \text{at } |r| \to \infty \end{cases}$$

Define Green's function as a solution of this PDE by replacing the source ρ by a delta function + homogeneous conditions.

$$\begin{cases} \nabla^2 G = -\delta(\vec{r}-\vec{r}\,') & (\rho \to \delta(\vec{r}-\vec{r}\,')) \\ G \to 0 \quad \text{as } |r| \to \infty & \text{Homogeneous bc's} \end{cases}$$

Want to show that $\phi(\vec{r}) = \iiint d\vec{r}\, \rho(\vec{r}\,')\, G(\vec{r}\,';\vec{r})$.

Proof:

$$\begin{cases} G\nabla^2\phi = -\rho(\vec{r})\, G(\vec{r};\vec{r}\,') \\ \phi\nabla^2 G = -\delta(\vec{r}-\vec{r}\,')\,\phi(\vec{r}) \end{cases} \Rightarrow \begin{aligned} G\nabla^2\phi - \phi\nabla^2 G = &-\rho(\vec{r})\,G(\vec{r};\vec{r}\,') \\ &+\delta(\vec{r}-\vec{r}\,')\,\phi(\vec{r}) \end{aligned}$$

Integrate:

$$\iiint d\vec{r}\, \underbrace{\left[G\nabla^2\phi - \phi\nabla^2 G \right]}_{\nabla\cdot(G\nabla\phi - \phi\nabla G)} = -\iiint d\vec{r}\, \rho(\vec{r})\, G(\vec{r};\vec{r}\,') + \underbrace{\boxed{\iiint \delta(\vec{r}-\vec{r}\,')\,\phi(\vec{r})\, d\vec{r}}}_{\phi(\vec{r}\,')}$$

integration by parts: $\oiint_{S_{|r|\to\infty}} ds\, \vec{n}\cdot(G\nabla\phi - \phi\nabla G) = 0$ since G and ϕ vanish at infinity

$$\Rightarrow \boxed{\phi(\vec{r}\,') = \iiint d\vec{r}\, \rho(\vec{r})\, G(\vec{r};\vec{r}\,')}$$ (one may switch $\vec{r}$ and $\vec{r}\,'$)

Exercise: Show that $G(\vec{r};\vec{r}\,') = G(\vec{r};\vec{r}\,')$

$$\boxed{\phi(\vec{r}) = \iiint d\vec{r}\,'\rho(\vec{r}\,')\, G(\vec{r}\,';\vec{r}) = \iiint d\vec{r}\,'\rho(\vec{r}\,')\, G(\vec{r};\vec{r}\,')}$$

(! general result: we used only bc at infinity and def. of G)

Check if $\phi(\vec{r})$ satisfies PDE:

$$\nabla^2_{\vec{r}}\phi(\vec{r}) = \nabla^2_{\vec{r}} \iiint d\vec{r}\,'\rho(\vec{r}\,')\, G(\vec{r}\,';\vec{r}) = -\iiint d\vec{r}\,'\rho(\vec{r}\,')\,\delta(\vec{r}-\vec{r}\,') = -\rho(\vec{r})$$

BC for ϕ ? as $|r| \to \infty$ $\phi(\vec{r}) = \iiint d\vec{r}' \rho(\vec{r}') G(\underset{\downarrow \infty}{\vec{r}};\vec{r}')$

if $\rho(\vec{r})$ is reasonably admissible function (goes to zero at infinity ...) $\therefore$ the integrand goes to zero uniformly $\Rightarrow$ $\phi(\vec{r}) \xrightarrow[|\vec{r}| \to \infty]{} 0$

Non-linear PDE Poisson

$$\begin{cases} \nabla^2 \phi(\vec{r}) = -\rho(\vec{r}, \phi(\vec{r})) \\ \phi \to 0 \quad |r| \to \infty \end{cases} \qquad \phi(\vec{r}) = \iiint d\vec{r}' \rho(\vec{r}', \phi(\vec{r}')) G(\vec{r};\vec{r}')$$

$\downarrow$

Definition of G: $\begin{cases} \nabla^2 G = -\delta(\vec{r}-\vec{r}') \\ G \to 0 \quad |\vec{r}| \to \infty \end{cases}$ restatement of the problem from BV problem to INTEGRAL EQ. for ϕ

*) more robust numerically

*) allows to find approximation of the solution

(! Green's function always satisfies a linear equation.)

Derive $G(\vec{r};\vec{r}')$ $\begin{cases} \nabla^2 G = -\delta(\vec{r}-\vec{r}') \\ G \to 0 \quad |\vec{r}| \to \infty \end{cases}$

Observations:

(i) G depends only on $(\vec{r}-\vec{r}')$ (not on each variable separtely ...)

- define $\vec{R} = \vec{r} - \vec{r}'$: $\nabla^2_{\vec{r}} \equiv \nabla^2_{\vec{R}}$ (laplacian is invariant to translations)

$\delta(\vec{r}-\vec{r}') = \delta(\vec{R})$

(ii) G depends only on the magnitude $|\vec{r}-\vec{r}'|$ (Exercise)

$G = g(|\vec{r}-\vec{r}'|)$, take $\vec{r}' = 0$ $(r = |\vec{r}|)$

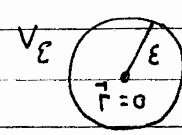
$G = g(r)$ G is spherically symmetric

$$\underbrace{\nabla^2 G = \frac{1}{r}\frac{d^2}{dr^2}(rG)}_{\text{spherically symmetric}} \quad \text{in fact} \quad \frac{1}{r}\frac{d^2}{dr^2}(rg) = -\delta(\vec{r})$$

$\vec{r} \neq 0$: $\frac{1}{r}\frac{d^2}{dr^2}(rg) = 0 \Rightarrow rg = A_1 r + A_2 \Rightarrow g(r) = A_1 + \frac{A_2}{r}$

BC at $r \to \infty$: $g \to 0 \Rightarrow A_1 = 0$ $\boxed{g(r) = \frac{A_2}{r}}$

Find A_2: $\iiint_{V_\varepsilon} d\vec{r}\, \nabla^2 G = -\iiint_{V_\varepsilon} \delta(\vec{r})\, d\vec{r}$



$$\Rightarrow \oint_{S_\varepsilon} ds\, \underbrace{\hat{n}\cdot\nabla G}_{\frac{\partial G}{\partial r} = \frac{dg}{dr}} = -1 \Rightarrow \left.\frac{dg}{dr}\right|_{r=\varepsilon} 4\pi\varepsilon^2 = -1$$

$$\Rightarrow \quad -\frac{A_2}{\varepsilon^2} 4\pi\varepsilon^2 = -1$$

So, we derived: $\boxed{A_2 = \frac{1}{4\pi}}$

$$G = g(|\vec{r}-\vec{r}'|) = \frac{1}{4\pi|\vec{r}-\vec{r}'|} \quad \text{(singular at } \vec{r}=\vec{r}'\text{)}$$

! The Green's function is a weak solution of the PDE
i.e. satisfies the integral form of the PDE

! Green's function is singular for dim > 1

Example 2 vibrating string under forcing

$$\underbrace{u_{tt} - c^2 u_{xx}}_{\text{wave eq.}} = \underbrace{-\mu u_x^2 u_{xx}}_{-g \text{ source term}}, \quad u = u(x,t) \quad \begin{cases} 0 < t < \infty \\ -\infty < x < \infty \end{cases}$$

for wave eq. the well-posed problem is the Cauchy problem

$u(x,0) = a(x)$

$u_t|_{t=0} = b(x)$ Non-linear PDE + non-homogeneous conditions

u, u_t

Steps:

(I) Write homogeneous PDE ($g=0$) + given conditions; solve it

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 \\ u(x,t=0) = a(x) \\ u_t(x,t=0) = b(x) \end{cases} \rightarrow \text{solution D'Alembert}$$

$$u_h(x,t) = \frac{a(x+ct) + a(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} d\xi\, b(\xi)$$

(II) Find a particular solution of the PDE with $g \neq 0$:
the particular solution has to satisfy the PDE but with homogeneous BC's
→ Use superposition via Green's function:

$$u_p(x,t) = \int dt' \int dx'\, G(x,t;x',t')\, g(x,t)$$

Define G:

$g = \delta(x-x')\delta(t-t')$

$$\begin{cases} G_{tt} - c^2 G_{xx} = -\delta(t-t')\delta(x'-x') \\ G(x,t=0) = 0 \\ G_t(x,t=0) = 0 \end{cases}$$

(III) At the end we want to write: $u = u_h + u_p$