$$\begin{pmatrix} -\beta & -\alpha \\ \alpha & \gamma\beta \end{pmatrix}\begin{pmatrix} \ell_1 \\ \ell_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} \alpha & \gamma\beta \\ \beta & \alpha \end{pmatrix}\begin{pmatrix} -\ell_1 \\ \ell_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

non-trivial solution $\Rightarrow$ $\alpha^2 - \beta^2\gamma = 0$ $\Rightarrow$ $\alpha = \pm\beta\sqrt{\gamma}$

Without loss of generality let's take $\beta = 1$ $\Rightarrow$ $\alpha = \pm\sqrt{\gamma}$
we find that the direction of the CHAR
on which superposition of the solutions = const is given by $\pm\sqrt{\text{speed}}$

February 25, 2004 Lecture 7

Pick up:
- Handout 5
- Hmwk 2
- Solution to Hmwk 1
- Practice set 2

Review session on Friday 4-5pm

PDE systems: THEME . PDE $\Rightarrow$ ODE(s)

Ex. $u_{tt} - \gamma u_{xx} = 0$ $\underset{\substack{v = u_x \\ w = u_t}}{\Longleftrightarrow}$ $\begin{cases} v_t - w_x = 0 \\ w_t - \gamma v_x = 0 \end{cases}$

Matrix form: $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} v_t \\ w_t \end{pmatrix} + \begin{pmatrix} 0 & -1 \\ -\gamma & 0 \end{pmatrix}\begin{pmatrix} v_x \\ w_x \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

Steps:

1 Define CHAR: $\begin{cases} \delta x = \alpha\varepsilon \\ \delta t = \beta\varepsilon \end{cases}$

2 Combine PDE sys. into 1 eqn. $\ell_1(v_t - w_x) + \ell_2(w_t - \gamma v_x) = 0$ any ℓ_1, ℓ_2
Read last equation as a statement about variations $\delta v, \delta w$ along CHAR

Eq. II : $m_1\delta v + m_2\delta w = 0 = m_1(v_x\alpha + v_t\beta) + m_2(w_x\alpha + w_t\beta)$

Eq. I : $\ell_1(v_t - w_x) + \ell_2(w_t - \gamma v_x) = 0$

Compare coefs. of v_t, v_x, w_t, w_x to find m_1, m_2, ℓ_1, ℓ_2

$\Rightarrow \begin{cases} \ell_1 = m_1\beta \\ -\ell_1 = m_2\alpha \\ \ell_2 = m_2\beta \\ -\gamma\ell_2 = m_1\alpha \end{cases}$ $\quad (\ell_1 \ \ell_2)\begin{pmatrix} \beta & \alpha \\ \alpha & \beta\gamma \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ $\Rightarrow$ $\begin{matrix} \beta\ell_1 + \ell_2\alpha = 0 \\ \alpha\ell_1 + \beta\gamma\ell_2 = 0 \end{matrix}$

$\Rightarrow \begin{vmatrix} \beta & \alpha \\ \alpha & \beta\gamma \end{vmatrix} = 0 \Longleftrightarrow \alpha = \pm\sqrt{\gamma}$ if $\beta = 1$

Find the general solution :

(i) $\alpha = \sqrt{\gamma}$, $\beta = 1$ ($\frac{dx}{dt} = \frac{\alpha}{\beta}$ gives the slope of the CHAR , we can take $\beta = 1$) the ratio matters

$$\begin{cases} \ell_1 = -\ell_2\,\alpha = -\sqrt{\gamma} \\ \ell_2 = 1 \end{cases} \qquad \begin{cases} m_1 = \ell_1 = -\sqrt{\gamma} \\ m_2 = \ell_2 = 1 \end{cases}$$

CHAR: $\frac{dx}{dt} = \sqrt{\gamma} \Rightarrow x - \sqrt{\gamma}\,t = C_1 = \text{const}$

$$-\sqrt{\gamma}\,dv + dw = 0 \Rightarrow -\sqrt{\gamma}\,v + w = K_1 = \text{const}$$

$(m_1 \delta v + m_2 \delta w = 0)$

we map constant to constant : $C_1 \overset{F}{\curvearrowright} K_1$

$$\boxed{\underbrace{-\sqrt{\gamma}\,v + w}_{K_1} = F(\underbrace{x - \sqrt{\gamma}\,t}_{C_1})} \qquad ①$$

(ii) $\alpha = -\sqrt{\gamma}$, $\beta = 1$

$$\begin{cases} \ell_1 = -\ell_2\,\alpha = \sqrt{\gamma} \\ \ell_2 = 1 \end{cases} \qquad \begin{cases} m_1 = \ell_1 = \sqrt{\gamma} \\ m_2 = \ell_2 = 1 \end{cases}$$

CHAR: $\frac{dx}{dt} = -\sqrt{\gamma} \Rightarrow x + \sqrt{\gamma}\,t = C_2 = \text{const}$

$$\sqrt{\gamma}\,dv + dw = 0 \Rightarrow \sqrt{\gamma}\,v + w = K_2 = \text{const}$$

$$\Rightarrow \boxed{\sqrt{\gamma}\,v + w = G(x + \sqrt{\gamma}\,t)} \qquad ②$$

Equations ① & ② are true at the same time. We have two arbitrary functions but also two unknowns v, w

$$① + ② \Rightarrow w = \frac{1}{2}\left[F(x - \sqrt{\gamma}\,t) + G(x + \sqrt{\gamma}\,t)\right] = u_t \qquad ③$$

$$② - ① \Rightarrow v = \frac{1}{2\sqrt{\gamma}}\left[G(x + \sqrt{\gamma}\,t) - F(x - \sqrt{\gamma}\,t)\right] = u_x \qquad ④$$

integrate ③ with time : $u(x,t) = \frac{1}{2\sqrt{\gamma}} \left[- \int^{x-\sqrt{\gamma}t} d\xi F(\xi) + \int^{x+\sqrt{\gamma}t} d\xi G(\xi) \right] + \underbrace{H(x)}_{\text{arbitrary}}$

integrate ④ with x : $u(x,t) = \frac{1}{2\sqrt{\gamma}} \left[- \int^{x-\sqrt{\gamma}t} d\xi F(\xi) + \int^{x+\sqrt{\gamma}t} d\xi G(\xi) \right] + \underbrace{\mathcal{Z}(t)}_{\text{arbitra}}$

$\Rightarrow \quad H(x) = \text{const} = \mathcal{Z}(t)$

Therefore, the general solution is $\boxed{u(x,t) = M(x - \sqrt{\gamma}t) + N(x + \sqrt{\gamma}t)}$

"General Theory"

PDE sys. $A_{ij} \underbrace{\frac{\partial u_j}{\partial t}}_{u_{j,t}} + a_{ij} \underbrace{\frac{\partial u_j}{\partial x}}_{u_{j,x}} + b_i = 0 \qquad i,j = 1,2,\dots n$

Step 1 : define CHAR $\begin{cases} \delta x = \alpha \varepsilon \\ \delta t = \beta \varepsilon \end{cases}$ (we have to find α, β ; unknown)

Step 2 : make linear combination of the equations

Ⓘ $\quad l_i A_{ij} u_{j,t} + l_i a_{ij} u_{j,x} + l_i b_i = 0 \quad | \, \varepsilon$

$\underbrace{l_i \varepsilon A_{ij} u_{j,t} + l_i \varepsilon a_{ij} u_{j,x}} + l_i \varepsilon b_i = 0$

$\Downarrow$ we want this to be

$m_j \delta u_j$ (linear comb. of δu_j) then use Taylor series expansion

$$m_j \delta u_j = m_j \left(u_{j,t} \underbrace{\delta t}_{\beta\varepsilon} + u_{j,x} \underbrace{\delta x}_{\alpha\varepsilon} \right)$$

$\Rightarrow \quad m_j (u_{j,t} \beta \varepsilon + u_{j,x} \alpha \varepsilon) + \varepsilon l_j b_j = 0$

Ⓘⓘ $\quad \varepsilon m_j (u_{j,t} \beta + u_{j,x} \alpha) + \varepsilon l_j b_j = 0$

Compare Ⓘ and Ⓘⓘ - they have to be the same
so the coefficients have to be the same

$u_{j,t}$: $\ell_i A_{ij} = m_j \beta$

$u_{j,x}$: $\ell_i a_{ij} = m_j \alpha$

Eliminate m_j $\Rightarrow$ $\ell_i A_{ij} \alpha = \ell_i a_{ij} \beta$ $\Rightarrow$ $\boxed{\ell_i(\alpha A_{ij} - \beta a_{ij}) = 0}$ (A)

linear homogeneous equation for ℓ_i $\Rightarrow$ $\det(\alpha A_{ij} - \beta a_{ij}) = 0$ (A')

suppose $\beta \neq 0$; define $x = \frac{\alpha}{\beta}$ $\Rightarrow$ $\boxed{\det[a_{ij} - x A_{ij}] = 0}$

from this det. equation we can find $x = \frac{dx}{dt}$: direction of CHAR

Special case: $A_{ij} = \delta_{ij}$ unit matrix

$\Rightarrow$ x are the eigenvalues of a_{ij}

if a_{ij}, A_{ij} are function of u, x,t then $x = x(u,x,t)$

like in quasi-linear problem.

Definition:

- The system is called hyperbolic if solution of system (A) are n-independent vectors $\ell^{(K)} = [\ell_i^K]$ $\begin{smallmatrix} K=1,\dots n \\ i=1,\dots n \end{smallmatrix}$ and (A') has real solutions $(\alpha^{(K)}, \beta^{(K)}) \neq (0,0)$

CHAR exist in the real space (real slope) and they are independent

- If some $(\alpha^{(K)}, \beta^{(K)})$ are non-real $\Rightarrow$ system is elliptic
- If $\ell^{(K)}$ are not all independent but $(\alpha^{(K)}, \beta^{(K)})$ are all real $\Rightarrow$ system is parabolic

- if $A = I$, in order to have a hyperbolic system a has to have n real different eigenvalues : for example, a real symmetric, pos. def.
- if A is not singular, multiply both sides by A^{-1}, go to previous

Ex. (Gas Dynamics)

Compressible inviscid flow of gas

4 quantities caracterize a gas:

Density ρ, velocity u, pressure p, entropy S

Usually $p = p(\rho, S)$

↳ known

System of PDE's for gas dynamics:

$$\begin{cases} \rho_t + u\rho_x + \rho u_x = 0 \\ u_t + u u_x + \frac{1}{\rho} p_x = 0 \\ S_t + u S_x = 0 \end{cases} \quad + \; p = p(\rho, S) \Rightarrow p_x = \underbrace{\frac{\partial p}{\partial \rho}}_{a^2} \rho_x + \underbrace{\frac{\partial p}{\partial S}}_{b^2} S_x$$

$$\Rightarrow \begin{cases} \rho_t + u\rho_x + \rho u_x = 0 \\ u_t + \frac{a^2}{\rho}\rho_x + u u_x + \frac{b^2}{\rho} S_x = 0 \\ S_t + u S_x = 0 \end{cases}$$

$$\underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{A} \begin{pmatrix} \rho_t \\ u_t \\ S_t \end{pmatrix} + \underbrace{\begin{pmatrix} u & \rho & 0 \\ a^2/\rho & u & b^2/\rho \\ 0 & 0 & u \end{pmatrix}}_{a} \begin{pmatrix} \rho_x \\ u_x \\ S_x \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

because A is a unit matrix, the direction of the CHAR given by the eig(a)

"$\frac{dx}{dt}$" $= \underset{\text{speed}}{\varkappa} = \frac{\alpha}{\beta}$: eigenvalues of $[a_{ij}]$ (check a posteriori if $\beta \neq 0$, if we find 3 eigv then $\beta \neq 0$)

$$\begin{vmatrix} u-\varkappa & \rho & 0 \\ a^2/\rho & u-\varkappa & b^2/\rho \\ 0 & 0 & (u-\varkappa) \end{vmatrix} = (u-\varkappa)\left[(u-\varkappa)^2 - a^2\right] = 0 \Rightarrow \begin{cases} \varkappa = u \\ \varkappa = u \pm a \end{cases}$$

the eigenvalues give us speeds : if we move along the CHAR at these speeds we will see some quantities are constant

(i) Case 1 $\quad u = \varkappa$

use $l_i(\alpha A_{ij} - \beta a_{ij}) = 0 \qquad (l_1 \; l_2 \; l_3)\begin{pmatrix} 0 & \rho & 0 \\ a^2/\rho & 0 & b^2/\rho \\ 0 & 0 & 0 \end{pmatrix} = 0 \Rightarrow \begin{cases} l_1 = 0 \\ l_2 = 0 \\ l_3 = \text{arbitrary} \end{cases}$

meaning of l_1, l_2, l_3 : the numbers with which we multiplied the PDE's

$\Rightarrow S_t + u S_x = 0 \quad$ but $u = \frac{dx}{dt} \Rightarrow S_t + \frac{dx}{dt} S_x = dS = 0$

along the CHAR with slope u (i.e. if we are moving at speed u) the entropy is conserved