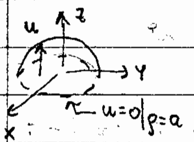
March 31, 2004 | Lecture 15

Opt. Reading: Eigenvalues probs: Garabedian Ch. 11; PDE classification: Kevorkian 4.1-4.4

Review: PDE $u_{tt} - c^2\nabla^2 u = 0$



Homogeneous bc's + principle of linear superposition

$$\Downarrow$$

$$u(\rho,\theta,t) = \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} J_n(K_{n,m}\,\rho)\left[A_{n,m}\cos(n\theta) + B_{n,m}\sin(n\theta)\right]\cos(\omega_{n,m}t)$$

$\longrightarrow K_{n,m}\,a = \gamma_{n,m}\ :\ J_n(\gamma_{n,m}) = 0$

$J_n(x)$ solution of Sturm-Liouville problem with ODE $(x \equiv \rho)$

$$\frac{d}{dx}\left(p\frac{d\phi}{dx}\right) + \left(q(x) + \mu s(x)\right)\phi = 0$$

$$K^2 = \frac{\omega^2}{c^2} \qquad x$$

Apply IC $u(\rho,\theta,t=0) = f(\rho,\theta)$ to find $A_{n,m}$ and $B_{n,m}$

IC: $$\sum_{n=0}^{+\infty}\sum_{m=1}^{+\infty} J_n(K_{nm}\rho)\left[A_{nm}\cos(n\theta) + B_{nm}\sin(n\theta)\right] = f(\rho,\theta)$$

$$\sum_{n=0}^{+\infty}\left\{\underbrace{\left(\sum_{m=1}^{\infty} J_n(K_{nm}\rho)A_{nm}\right)}\cos(n\theta) + \underbrace{\left(\sum_{m=1}^{\infty} J_n(K_{nm}\rho)B_{nm}\right)}\sin(n\theta)\right\} = f(\rho,\theta)$$

if we consider $\rho = \text{const}$, this is a standart Fourier series

$$\sum_{n=0}^{+\infty} C_n(\rho)\cos(n\theta) + D_n(\rho)\sin(n\theta) = f(\rho,\theta)$$

$$D_n(\rho) = \frac{1}{\pi}\int_0^{2\pi} d\theta\, f(\rho,\theta)\sin(n\theta) \quad : \text{known}$$

$$C_n(\rho) = \frac{1}{\pi}\int_0^{2\pi} d\theta\, f(\rho,\theta)\cos(n\theta)\ ,\ n \neq 0 \ : \text{known}$$

$$C_0(\rho) = \frac{1}{2\pi}\int_0^{2\pi} d\theta\, f(\rho,\theta)$$ (average value of f over the circle for fixed ρ)

Find A_{nm} and B_{nm}:

$$C_n(\rho) = \sum_{m=1}^{\infty} A_{nm} J_n(K_{nm}\rho)\ ,\qquad D_n(\rho) = \sum_{m=1}^{\infty} B_{nm} J_n(K_{nm}\rho)$$

The Bessel function $J_n(k_{nm}\rho)$ are solutions of a S-L problem ⇒ they form a discrete complete orthogonal set of functions + weight $s(x)$; we can expand functions on them.

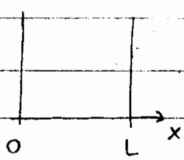
$$\int_0^a d\rho\, \underbrace{\rho}_{s(x)}\, J_n(k_{nm}\rho)\, J_n(k_{nm'}\rho) = \delta_{mm'}$$

$$A_{nm} = \frac{\int_0^a d\rho\, \rho\, J_n(k_{nm}\rho)\, C_n(\rho)}{\int_0^a d\rho\, \rho\, J_n(k_{nm}\rho)^2}, \qquad B_{nm} = \frac{\int_0^a d\rho\, \rho\, J_n(k_{nm}\rho)\, D_n(\rho)}{\int_0^a d\rho\, \rho\, J_n(k_{nm}\rho)^2}$$

⟶ easy to calculate

Back to "Eigenvalues problem"

Ex. Linear Schrödinger equation
$$\begin{cases} i\Psi_t = -\Psi_{xx} + \cancel{V(x,t)}^{\,0}\,\Psi \\ \psi(0,t) = \psi(L,t) = 0 \end{cases}$$



we find the solutions:

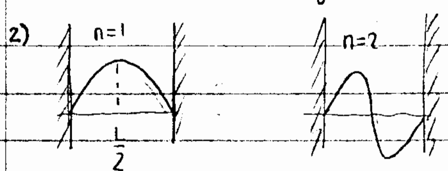
$\psi(x,t) = e^{-i\omega t}\varphi(x)$ where $\omega = \omega_n = \left(\frac{n\pi}{L}\right)^2$, $n = 1, 2, \ldots$

"eigenvalues"

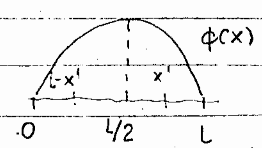
Remark:

1) the eigenvalues are calculated independently of $\varphi(x)$ i.e. what the amplitude is. (do not involve the values of $\varphi(x)$)
→ this will change for non-linear problems

2)



Lowest energy solution $n=1$:



the solution is symmetric /even around $L/2$. Do we expect this?

If $\psi(x')$ solves ODE, $\pm\phi(L-x')$ solves the ODE too.

for $n=1$: $+\Phi(L-x')$ is a solution
for $n=2$: $-\Phi(L-x')$ is a solution
(parity change)

→ there exists a lot of theories about LINEAR PDE's but it doesn't generalize to non-linear. (confusing)

~~Ex:~~ Non-linear Schrodinger equation

$$\begin{cases} i\Psi_t = -\Psi_{xx} + \overset{0}{\cancel{V(x,t)}}\,\Psi + a|\Psi|^2\Psi & 0<x<L, \quad a>0 \\ \Psi(0,t) = 0 = \Psi(L,t) & \text{(same BC's)} \end{cases}$$

the term "eigenvalues" should not be used for non-linear problems.

→ use natural frequencies

Does the PDE allows for a solution of the form

$$\Psi(x,t) = e^{-i\omega t}\phi(x) \qquad (\omega>0)$$

$$\Psi_t = -i\omega e^{-i\omega t}\varphi(x), \qquad \Psi_{xx} = e^{-i\omega t}\varphi''(x)$$

$$\Rightarrow \omega\varphi(x)\cancel{e^{-i\omega t}} = -\varphi''(x)\cancel{e^{-i\omega t}} + a|\varphi|^2\varphi\cancel{e^{-i\omega t}}$$

ODE for φ: $\begin{cases} \omega\varphi(x) = -\varphi''(x) + a|\varphi|^2\varphi(x) & 0<x<L \\ \varphi(0) = \varphi(L) = 0 \end{cases}$

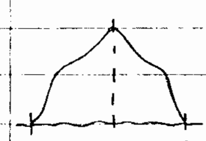
non-linear ODE, how do we solve it? → can be viewed as potential

Suppose ω is in some set $\{\omega_n\}_{n=0}^{+\infty}$.

→ assumption:

If $a \ll 1$, the solution of the NL problem should resemble the linear one.

look at $n=1$: we expect an even solution because the non-linear term doesn't change the symmetry



0 L/2 L → nice property: $\phi'(L/2) = 0$ and (assume also that) ϕ = real

⇒ solve in $(0, L/2)$

Solve the ODE:

φ' is missing → multiply by φ' and get full differentials

$$\underbrace{\omega\varphi\varphi'}_{\frac{1}{2}(\varphi^2)'} = -\underbrace{\varphi''\varphi'}_{\frac{1}{2}[(\varphi')^2]'} + a\underbrace{\varphi^3\varphi'}_{\frac{1}{4}(\varphi^4)'} \Rightarrow \omega\varphi^2 = -(\varphi')^2 + a\frac{1}{2}\varphi^4 + C$$

Apply condition at $L/2$ to get rid of φ':

at $x = L/2$ $\begin{cases} \varphi'(L/2) = 0 \\ \varphi(L/2) = \varphi_0 \text{ : unknown} \end{cases}$

$$\Rightarrow \quad \omega\varphi_0^2 = -0 + \frac{a}{2}\varphi_0^4 + C \quad \Rightarrow \quad \boxed{\omega\varphi_0^2 - \frac{a}{2}\varphi_0^4 = C}$$

Solve the first order ODE:

$$(\varphi')^2 = \frac{a}{2}\varphi^4 - \omega\varphi^2 + \omega\varphi_0^2 - \frac{a}{2}\varphi_0^4$$

$$= \omega(\varphi_0^2 - \varphi^2) - \frac{a}{2}(\varphi_0^2 - \varphi^2)(\varphi_0^2 + \varphi^2) = (\varphi_0^2 - \varphi^2)\left[\omega - \frac{a}{2}(\varphi_0^2 + \varphi^2)\right]$$

$$\frac{d\varphi}{dx} = \pm\sqrt{(\varphi_0^2 - \varphi^2)\left[\omega - \frac{a}{2}(\varphi_0^2 + \varphi^2)\right]}$$

$\hookrightarrow$ we choose + sign because the solution is increasing

$$\int_{\varphi(0)=0}^{\varphi(x)} \frac{d\varphi}{\left\{(\varphi_0^2 - \varphi^2)^{1/2}\left[\omega - \frac{a}{2}(\varphi_0^2 + \varphi^2)\right]\right\}^{1/2}} = \int_0^x dx = x + \text{const}$$

(but const goes away from BC at $x=0$)

$$x = \int_0^{\varphi(x)} \frac{d\xi}{(\varphi_0^2 - \xi^2)^{1/2}\left[\omega - \frac{a}{2}(\varphi_0^2 + \xi^2)\right]^{1/2}}$$

set $\xi = \varphi_0 \sin\psi \Rightarrow d\xi = \varphi_0 \cos\psi \, d\psi$, $\sqrt{\varphi_0^2 - \xi^2} = \varphi_0 \cos\psi$

$$x = \int_0^{\sin^{-1}(\varphi/\varphi_0)} \frac{d\psi}{\left[\omega - \frac{a}{2}\varphi_0^2(1 + \sin^2\psi)\right]^{1/2}} = \frac{1}{\sqrt{\omega - \frac{a}{2}\varphi_0^2}} \int_0^{\sin^{-1}(\varphi/\varphi_0)} \frac{d\psi}{\sqrt{1 - \underbrace{\left(\frac{a/2\,\varphi_0^2}{\omega - a/2\,\varphi_0^2}\right)}_{=\kappa^2}\sin^2\psi}}$$

Def:

$$\int_0^{\theta} \frac{d\psi}{\sqrt{1 - \kappa^2\sin^2\psi}} \equiv \mathbb{K}(\theta, \kappa)$$ incomplete elliptic integral of 1st kind

$\Rightarrow$ we get the solution : $\boxed{\mathbb{K}\left(\sin^{-1}\left(\frac{\varphi}{\varphi_0}\right), \kappa\right) = \sqrt{\omega - \frac{a}{2}\varphi_0^2}\; x}$

to get φ, we need to take the inverse of $\mathbb{K}$

Def: $\mathbb{K}(\sin^{-1} f, \kappa) = \xi \quad \Rightarrow \quad f = \mathrm{sn}(\xi; \kappa)$: elliptic function

$\Rightarrow$ the solution is $\boxed{\varphi(x) = \varphi_0 \cdot sn\left(\sqrt{\omega - \frac{a}{2}\varphi_0^2}\cdot x \, ; \kappa\right)}$

take $x = L/2$

$$\varphi(L/2) = \varphi_0 = \varphi_0 \; sn\left(\sqrt{\omega - \frac{a}{2}\varphi_0^2}\,\frac{L}{2} \, ; \kappa\right)$$

$$\Downarrow$$

$$1 = sn\left(\sqrt{\omega - \frac{a}{2}\varphi_0^2}\,\frac{L}{2} , \kappa\right)$$

$$\Downarrow$$

we can get ω but def of ω depends on φ_0, the value at the pick

April 5, 2004 — Lecture 16

Pick up: Handouts 7, 8, Solutions to Hmwk 3
Hmwk 4, Problem Set 4

Pick up: Graded Test 4-6 pm

Opt. Reading: Kevorkian 4.1-4.4, Debnath 1.11

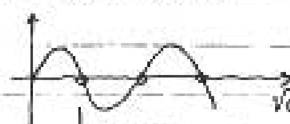
Natural frequencies:

Ex 1 Linear Schrödinger eqn. (LSE)

$$\begin{cases} i\Psi_t = -\Psi_{xx} + \cancel{V(x,t)}^{\,0}\,\Psi \\ \Psi(0,t) = 0 = \Psi(L,t) \end{cases}$$ Homogeneous problem

Try $\Psi(x,t) = e^{-i\omega t}\varphi(x) \Rightarrow \begin{cases} \varphi''(x) + \omega\phi(x) = 0, & 0 < x < L \\ \phi(0) = 0 = \phi(L) \end{cases}$

$\Rightarrow \phi(x) = \underbrace{A}_{\text{arbitrary constant}} \sin(\sqrt{\omega}x)$



$\sqrt{\omega}x$

Zeros correspond to $\omega_n = \left(\frac{n\pi}{L}\right)^2$ $n = 1, 2, \ldots$

ω_n ! independent of A

constraint does not come from the homogeneous equation $\rightarrow$ Const. A : $\int_0^L dx\, |\Psi|^2 = 1 \Rightarrow |A|^2 = \frac{2}{L}$

Ex. 2 Nonlinear Schrödinger eqn. (NLSE)

$$\begin{cases} i\Psi_t = -\Psi_{xx} + a|\Psi|^2\Psi \\ \Psi(0,t) = 0 = \Psi(L,t) \end{cases}$$

Try $\Psi = e^{-i\omega t}\phi(x) \Rightarrow \begin{cases} \omega\phi(x) = -\phi''(x) + a|\phi|^2\phi, & 0<x<L \\ \phi(0) = 0 = \phi(L) \end{cases}$

"we try to find natural frequencies"