18.310 Homework # 8

1a: Suppose that you have a necklace of length 15, and three colors of beads. How many different patterns are there for stringing the beads on the necklace?
1b: Suppose that you use five beads each of the colors green, blue, and yellow? How many different patterns are there to string these beads on the necklace?
1b: Suppose that you use three yellow beads, 4 blue beads, and 8 green beads. Now how many different patterns are there to string the beads on the necklace?

2: In the proof of the tree counting theorem, we showed a correspondence between functions from $\{1, 2, \ldots, n-2\}$ to $\{1, 2, \ldots, n\}$ and labeled trees. We can use this to prove theorems about these functions.
2a: Let the smallest label of a vertex involved in a cycle be i. In a random function as above, show that the expectation of the ratio of the number of vertices in the cycle containing i and the total number of vertices involved in cycles is larger than $\frac{1}{2}$ (consider for the probability space only functions from $\{1, 2, \ldots, n-2\}$ to $\{1, 2, \ldots, n\}$ which have cycles).
2b: Assume that in a random labeled tree, the number of vertices on the path from $n-1$ to n is l. What is the expected number of cycles in the corresponding random permutation? Make sure your method is accurate enough to find the constant on the highest-order term.
Hint: let I_i be an indicator function which is 1 if the i^{th} ranked vertex on this path is closer to $n-1$ than any of the vertices on the path with labels less than it.

Note: The argument that we used in the tortoise-and-the-hare factoring algorithm to show that the length of the cycle in that algorithm $O(\sqrt{n})$ can be combined with part (2a) to show that the expected number of vertices in a random tree on the path from $n-1$ to n is $O(\sqrt{n})$. This, together with (2a) and (2b) gives a lot of information about the typical structure of random functions, as well as the shape of random labeled trees.

3: Consider the sequence 1, 3, 8, 20, 48, ..., where $S_n = 4 * S_{n-1} - 4 * S_{n-2}$. Use generating functions to figure out a formula for S_n.

4: Let T_n be the number of ways of tiling a $2 \times n$ strip of squares by tiles which are of size either 1×2 or 1×4? Give a recurrence which expresses T_n in terms of T_i, with $i < n$. Now, write down a polynomial so that T_n can be expressed as

$$\sum_i \alpha_i \lambda_i^n$$

where α_i are constants and λ_i are the roots of the polynomial. On a spreadsheet, compute T_n for $n = 1$ through 40. Estimate the largest root of the polynomial by taking the ratio T_n/T_{n-1}. How close are you?