15. Planarity and Coloring

15.1 Introduction

We have been considering the notions of the colorability of a graph and its planarity (see notes 14).

We have seen that a graph can be drawn in the plane if and only if it does not contain a subdivision of K_5 or $K_{3,3}$. We have also seen how to determine whether the coloring number of a graph is 2. We shall now examine the much harder problem of characterizing graphs of higher coloring number (at least 3-colorable).

A natural question, which was raised back in the 19^{th} century is: *What is the largest coloring number among planar graphs? In other words, what is the largest number of colors you need to color any graph that can be drawn in the plane?*

A man named Kempe published a proof in 1879 that all planar maps are 4-colorable. However, this proof was incorrect and ten years later, someone noticed this and pointed it out.

As a result, this became a well-known problem. Was Kempe right? Is the largest coloring number among planar graphs 4?

Many mathematicians worked very hard on this problem and produced many partial results. Finally, about a hundred years after Kempe's paper, a computer-aided proof was announced by Appel and Haaken. It uses few ideas beyond those employed by Kempe, but applies them to thousands of cases using a computer, and provides a proof of his claim.

We will provide a proof that five colors are enough to color any planar graph, and then present Kempe's flawed proof that four colors are enough.

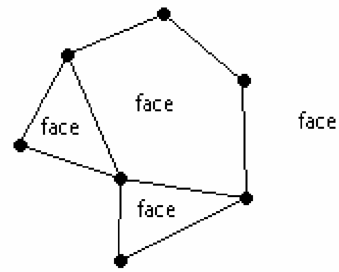
First, however, we must prove a few results that will be necessary in our later results.

15.2 Euler's Formula

Suppose we have a planar graph G, and we make a drawing of it as we discussed in notes 14. The drawing itself has properties that G, as a set of edges and vertices, does not possess.

In particular, the edges of G divide the plane into separated regions called **faces**. For example, if we have a simple 3-cycle as our graph, then it has two faces, the "inside" and the "outside". The following graph has 4 faces (3 inside and one outside).

(1)



The **boundary** of a face is the set of edges that enclose it.

If we define the parameters *v*, *e*, and *f* to be the number of vertices, edges, and faces of the drawing of G, there is a wonderful relation between these numbers that holds whenever G is a connected graph:

$$v - e + f = 2$$

If G has more than one connected component, then we change the formula a little, with *cc* being the number of connected components of G:

$$v - e + f = 1 + cc$$

This relation is called **Euler's formula.** We will now prove it.

It is obvious that this relation holds for a single vertex with no edges, as well as for two vertices with one edge between them. We shall use an inductive proof by assuming that Euler's relation holds for a graph, and showing that it still holds if we add a new vertex or a new edge.

Suppose we have a planar drawing of a graph G and we add a new edge or a new vertex to G so that the new drawing of the resultant graph has no edge crossings and is thus still planar.

There are only a small number of ways in which we could add a vertex or an edge.

If we add a new vertex on an edge of G, then this splits that edge into two edges, so *e* and *v* both increase by 1, so the equality holds. If we add a vertex not on any edge, then it is an isolated point and so *v* and *cc* increase by 1, and still the equality holds. These are the only ways in which on can add a vertex, since it is either on an edge or not. Thus, Euler's formula holds for the addition of a vertex.

Suppose we add an edge between two vertices of G. If there is a path in G between the end of the new edge, then adding the edge creates a face and so *e* and *f* increase by 1. Since there was already a path connecting the vertices, *cc* remains unchanged. If there was no path connecting them, then *f* remains unchanged, but *e* increases by 1 and *cc* decreases by 1 (since we have connected two previously unconnected components with this new edge). Since either a path connected the vertices of the new edge or it did not, so these are the only ways to add a new edge. Since both of them preserve Euler's formula, we see that it holds for the addition of an edge.

We see that Euler's formula holds for the addition of a vertex or an edge to G, so we see by induction that Euler's formula is true.

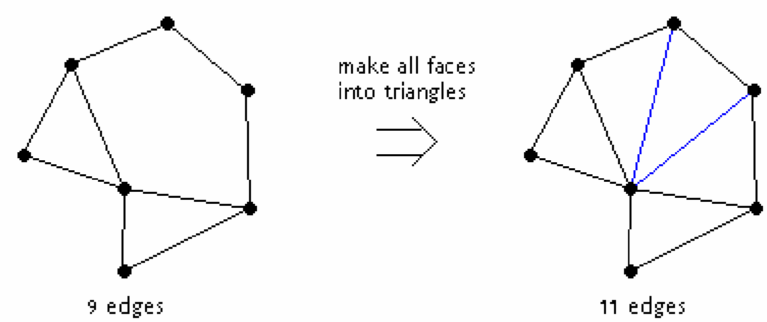
□

15.3 The Average Degree of a Planar Graph

We now ask: *what is the largest number of edges that a planar graph with* v *vertices can have?*

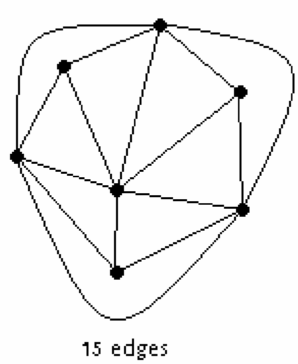
We can deduce the answer from Euler's formula.

Each face defined by a drawing of G in the plane is bounded by a cycle of G. If that cycle is not a triangle, we can add an edge between two opposite vertices and increase the number of edges. Let us look at graph (1) from above as an example:



The above illustration only shows how to turn one face into all triangles. If we really want to maximize the number of edges in graph (1), then we can make 4 more triangles in the following way (note that triangles in this sense are faces with 3 edges; the edges do not have to be straight):

(2)



We conclude then that a graph G on *v* vertices with the most edges will have triangles for all its faces.

On such a graph, each face has three edges on its boundary. We will form "edge-face pairs" between a face and each of its edges. This means there are 3 edge-face pairs for each face, and thus $3f$ of them altogether.

We can also see that each edge bounds two faces, so the number of edge-face pairs will also be $2e$.

We deduce then that $f = 2e/3f$, so that in this case we can write Euler's formula as[1]

$$v - 3 + 2e/3 = 2$$

or

$$3v = e + 6$$

If we look at graph (2) above, we can see that this last equality holds in this case.

We notice also that since each edge has two vertices, the number of edge-vertex pairs is $2e$ and is also equal to the sum of the degrees of all the vertices (recall that the degree of a vertex is the number of edges that contain it as an endpoint).

This tells us that the sum of the degrees of the vertices of any planar graph G with v vertices and the maximum number of edges is $2e$. Using the equation $3v = e + 6$, we get that:

[The sum of the degrees of the vertices of G] = $6v - 12$

If we divide this by v, we get that

[The average degree of a vertex of G] = $6 - 12/v$

We conclude then that for any planar graph on v vertices:

$e \leq 3v - 6$

and

The average degree of a vertex is strictly less than 6

It follows from this second statement that:

We can always find a vertex of degree 5 or less in any planar graph

[1] Since we are maximizing the number of edges, this will be a connected graph and thus cc = 1, and we can use the formula v – e +f = 2.

15.4 The Five Color Theorem

We now prove that we can color any planar graph with 5 colors.

We proceed by induction, and assume that any graph with fewer vertices than G can be colored with 5 colors.

If G possesses a vertex *v* of degree 4, then we look at the graph G – {*v*}, which we use to denote the graph consisting of G without the vertex *v*. By our inductive hypothesis, this graph can be colored with 5 colors. Now we add *v* to G – {*v*}. The neighbors of *v* will have colors say A, B, C, and D. But we have 5 colors at our disposal, so we can color *v* with the fifth color, say E, without causing any problems. This means that we can extend our coloring of G – {*v*} to G, and thus G is 5-colorable.

The same argument applies if G has a vertex with degree less than 4. So let us assume that all vertices of G have degree at least 5. We know from the last section that at least one vertex of G must have degree 5.

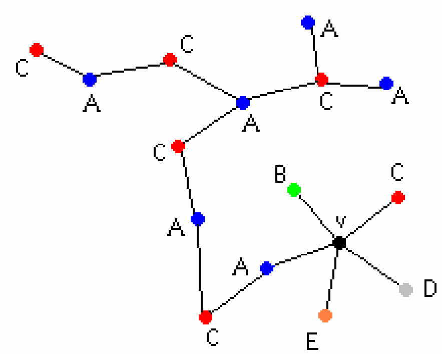
So suppose *v* has degree 5. Once again we will consider G – {*v*}, and color it with 5 colors.

We will be able to extend the coloring to *v* unless all its neighbors have different colors.

Suppose then that the neighbors of *v* in G have colors A, B, C, D, and E in order.

Now, we start from the neighbor vertex with color A and find all its neighbors of color C, and all their neighbors of color A, and all their neighbors of color C, until all the vertices that can be reached in this way have been found. We will call this an "AC chain". What we get will look something like this:

(3)

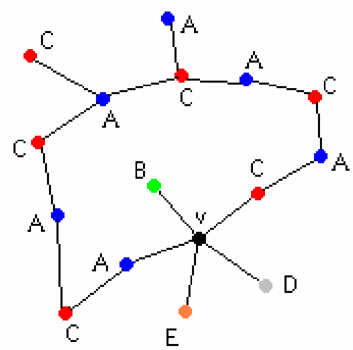


Assume that, like in graph (3), that the path of alternating C and A vertices does not lead back the C colored vertex that is a neighbor of v. This allows us to take all the vertices in this path and reverse colors A and C. This will not harm the coloring of the graph G – {*v*}, because it only affects the vertices on the AC chain, and there are no vertices of color A or C not on the chain that share an edge with any vertex on the chain because of the way we made it. This means we could not possibly have two neighbor vertices of the same color that were not present in the graph previously, which means there are none since G – {*v*} is 5-colorable.

The effect of this switch is that now *v*’s neighbor that used to be color A is now color C. So *v* has two neighbors of color C (since the C vertex was unaffected by the switch since it was not on the path) and no neighbor of color A. Thus we can extend the coloring of G – {*v*} to G by coloring *v* with color A. This means that G must be 5-colorable.

Thus, we suppose that there is an AC chain like the one described, except that it connects the neighbors of G of colors A and C:

(4)

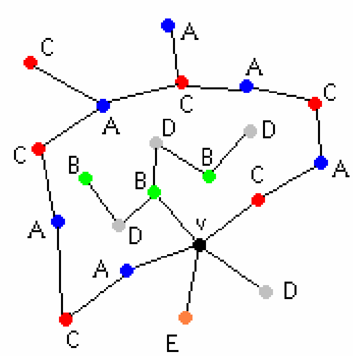


We cannot use the same trick as before, because if we reverse A and C on the path, then *v* will still have a neighbors of colors A and C, just in opposite places.

Notice, however, that adding v to this chain creates a cycle in G, and that the neighbor of G of color B is on one side of this cycle while the neighbors of colors D and E are on the other side of it (this is independent of how we draw the cycle, and is not just true in graph (4)).

This means that if we start at the B neighbor of *v* and make a BD chain, there is no way that this path could contain any vertices outside of the aforementioned cycle. This is because if the chain only contains B and D vertices, then it cannot contain any of the vertices of the cycle, and thus it is trapped inside the cycle. This guarantees that the BD chain will not be connected to *v*’s neighbor of color D, which lies outside the cycle:

(5)



So we can reverse the colors B and D on the BD chain and since it is not connected to *v's* D colored neighbor, this means that there will be two neighbors of color D, and none of color B. Thus, we can extend the coloring to all of G by giving *v* the color B.

Therefore, we see how we can color any planar graph with 5 colors.

□

The AC and BD chains that we used in this proof are called **Kempe chains**.

15.5 Kempe's False Proof of the Four Color Theorem

Here is the way Kempe tried to prove the 4 color theorem.

Suppose, again by an induction hypothesis, that G – {*v*} can be colored in 4 colors, where *v* has 5 neighbors in G.[2] We can extend the coloring to *v* unless all 4 colors appear on its neighbors. This means that starting at some neighbor, the neighbor colors in order are A, B, A, C, D.

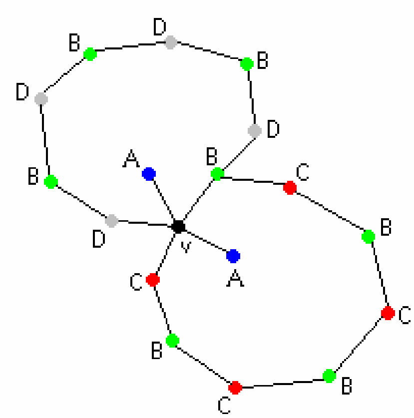
We will be able to extend the coloring to *v* if there is no Kempe chain linking the vertices colored B and C, by switching the colors of B and C on all vertices of the chain starting at B.

So there must be such a chain, namely a B and C colored path between the neighbors of *v* of colors B and C.

By an identical argument, there must also be a B and D colored path between the neighbors of *v* of colors B and D, or again we could extend the coloring to *v* by switching colors.

This gives us two cycles isolating the two vertices of color A from each other:

(6)



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[2] If v has 4 neighbors, then we can use the exact same argument as the previous section to show that G is 4-colorable. If *v* has less than 4 neighbors, then we can trivially show that G is 4-colorable.

Because of the BC chain, we can make an AD chain starting with *v*'s second A colored neighbor that is trapped and cannot reach *v*'s D colored neighbor. Thus, we can switch the colors A and D on the AD chain without changing the D colored neighbor at all.

Furthermore, because of the BD chain, we can similarly form an AC chain using *v*'s first A colored neighbor that cannot reach *v*'s C colored neighbor. Thus, we can switch the colors A and C on this chain without affecting the C colored neighbor.

So, Kempe argued, do both of these things. This will change the first A to C and the second A to D, leaving us free to color *v* with color A, which will extend the coloring to G and prove the theorem.

As already noted, this proof was published and not questioned for about 10 years.

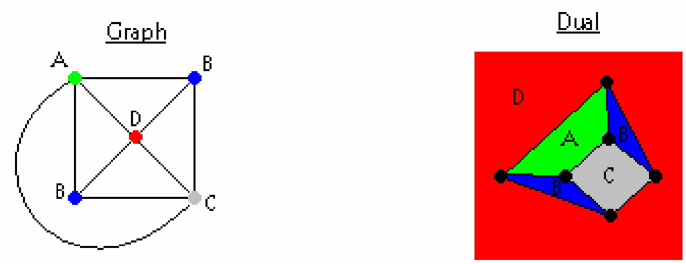
Can you see the flaw in it?

15.6 Another Formalization of the Four Color Problem

If a graph G is 4-colorable, then it is clear that removing any edges from G will not affect this. So, if we could prove that for *v* the planar graph on *v* vertices with the maximum number of edges (as discussed in 15.3) was 4-colorable, it would mean that any graph on *v* vertices was 4-colorable.

Suppose we have such a maximum-edge graph that we can color with 4 colors; A, B, C, and D. Now, we are going to form a new graph G' by switching the faces and vertices of G in the following way: We assign a vertex to each face of G, and we make an edge between two vertices if the faces corresponding to those vertices share an edge on G. This means that each face of G' is a vertex in G, and we color these faces the same as the corresponding vertex of G. We call G' the **dual** of G. Here is an example of a graph and its dual (the numbers correspond to color numbers we will use later):

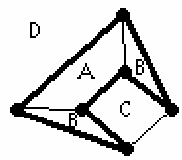
(7)



The red box in our drawing of the dual above corresponds to the red vertex in the original graph.

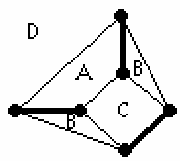
Since all the faces of a maximum-edge graph are triangles (as we saw in 15.3), this means that all the vertices of the dual to this graph will have degree 3. You can see that this is true in the graph (7) above.

We now look at the edges of G' which form the boundary between the faces that are color A and color B and the faces that are color C and color D. Since each vertex is at the apex of three adjacent faces and hence 3 colors, this boundary will pass through each vertex. We draw the dual of graph (7) below with these boundaries darkened below (we have chosen not to color it in order to make the boundaries easier to see):



AB cs. CD boundary

The edges of G' which are not boundary edges will have degree 1 at each vertex. It will therefore be a matching (or pairing) of the vertices of this graph:



Had we looked at the edges not in the A C vs. B D boundary we would have gotten a different matching, and had we looked at the edges not in the A D vs. B C boundary we could get a third matching, and all of these are disjoint. (Try drawing these for the above graph to verify for yourself).

Since each of the matchings are disjoint, and their union is the edges of G', we see that the union of the matchings for two of the color sets is a boundary for the third. It follows then that the boundary edges in each case will form an even cycle, which connects all the vertices of G'. This is true for the dual to any maximum-edge planar graph.

This tells us that a maximum-edge planar graph will have a four coloring if its dual has a collection of even cycles which pass through all its vertices exactly once.

Many attempts to prove the four color theorem have been based on this viewpoint. In particular, you could prove the four color theorem if you could show that every planar graph, with all its vertices of degree three, has a single cycle that passes through each of its vertices exactly once.

Such a cycle is called a **Hamiltonian cycle** and these have been the subject of much study. Unfortunately, it has been shown that there are planar graphs with all vertices of degree three that do not have Hamiltonian cycles.

15.7 Brook's Theorem

There is a nice result that we can prove about coloring graphs which can provide an upper bound for the coloring number of a large subset of graphs. The result, called **Brook's Theorem,** is as follows:

Suppose the maximum degree of a connected graph G is d. Then we can color the vertices of G in d colors, unless G is a complete graph or an odd length chordless cycle, which both require d + 1 colors.

Here is a proof of the theorem.

We have already seen in notes 14 that the complete graph on $d + 1$ vertices, in which every vertex has degree d, requires $d + 1$ colors. We have also seen that an odd length chordless cycle, whose maximum vertex degree is 2, requires 3 colors. We just have to show that all graphs not of this type can be colored with d colors.

Let G be a graph of maximum degree d, which is not a complete graph or an odd length chordless cycle. Since G is not complete, there are pairs of vertices that do not share an edge and since the graph is connected there must be two vertices whose distance in the graph from each other is 2. (The **distance** between two vertices on a connected graph is the number of edges in the shortest path between them). Call these vertices v_1 and v_2 and let v_n be a common neighbor (a vertex of distance 1 from each).

We assign the vertices indices in order, starting at v_1 and v_2 and ending at v_n in such a way that each vertex v_j other than v_n has at least one edge joining v_j with a vertex of higher index than j. (This ordering may or may not be possible).

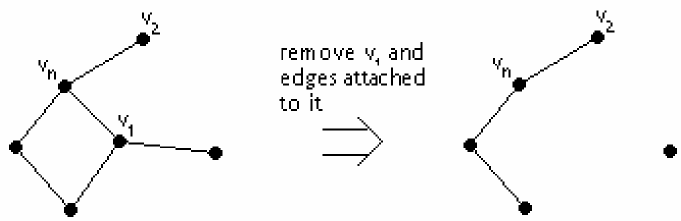
If the graph G' obtained by removing v_1, v_2 and all the edges attached to them from G is connected, we can obtain an ordering of the vertices by starting at v_n, then making v_{n-1} a neighbor of it, v_{n-2} a neighbor of v_n or v_{n-1}, and so on until all the vertices are listed.

If we can do this, we can color G with d colors as follows:

Give v_1 and v_2 the same color. Go through the rest of the graph in order of vertex index. For each v_j, at least one of its neighbors has higher index and has not yet been colored, and since its degree is at most d, a maximum of $d – 1$ neighbors have been colored. We therefore have a color left for v_j that does not appear on a neighbor and color v_j with that color. When we get to v_n, two of its neighbors, v_1 and v_2 have the same color, so again at most $d – 1$ colors appear on its neighbors and you can color it. In this way, we color G with d colors.

We will not be able to find the desired ordering of the vertices if v_1 and v_2 are such that by removing one (or both) of them we separate the resulting graph into disconnected pieces. Here is an example of a graph where this happens:

(8)



(Notice that if we tried to number the vertices of graph (8) in the manner above, we not be able to because the rightmost vertex neighbor of v_1 cannot have a neighbor that is numbered higher than it. We could remedy this by adding an edge between this vertex and the bottommost vertex, but then the removal of v_1 would no longer disconnect the graph.)

We will show to how color the graph with *d* colors for cases where numbering the vertices does not work.

We assume that *d* is at least 3, because if G is not a 2-cycle or a path then *d* is at least 3 and if G is a path or a 2-cycle then it is apparent that a suitable coloring exists.

We also assume by induction that Brook's theorem holds for every subgraph of G.

If the graph G' obtained from G by removing v_1 (or v_2) is not connected, then we can color v_1 and each of the connected components of G' separately in *d* colors by using our induction hypothesis, and glue them together to color G.[3]

It could be that the graph obtained by the removal of v_1 or v_2 by themselves is still connected, but the graph obtained by removing both v_1 and v_2 is disconnected. If so, we can color each of the connected components and v_1 and v_2 (with an additional edge connecting them) in *d* colors by induction, and then glue them together to color G.

This coloring can fail only if one of the graphs for a connected component is complete with degree *d* and the other is the triangle containing v_1, v_2, and v_n. In that case, it can still be easily colored using *d* colors.

This proves Brook's Theorem

□

~Edited by Jacob Green

[3] Now, it is possible that one of the connected components of G' is a complete graph or an odd cycle. However, if we look at the vertex *v* where v_1 is attached to this connected component in G, then the degree of *v* will be greater than the maximum degree of the connected component itself, and thus G is colorable with *d* colors (since our inductive hypothesis and Brook's theorem tell us that they can be colored by (1 + their maximum degree) colors.