14. Some Graph Theory

**Editor's Note: These notes are quite dense and pack a lot of material in. If you have trouble following them, you may wish to reread the section or consult an outside source.*

14.1 Introductory Definitions

We will investigate some of the basics of graph theory in this section.

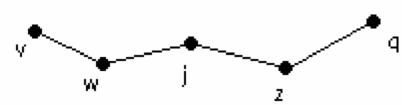
A **graph** G is a collection of vertices and edges connecting some or all of these vertices. More formally, a graph is a collection E of distinct unordered pairs of distinct elements of a set V. The elements of V are called **vertices** or nodes, and the pairs in E are called **edges** or arcs or "the graph". (If a pair (w, v) can occur several times in E, we call the structure a **multigraph**. If edges like (v,v), which are called loops, are allowed, it is called a "graph with loops".)

Graphs are things that underlie many mathematical structures, and in fact anything that involves pairs of elements, which includes any kind of relationship between pairs of individuals.

In a graph, a **path** from vertex v_1 to vertex v_2 is a sequence of edges such that the first edge contains v_1, the last edge contains v_2, and each consecutive pair in the sequence has a vertex in common. The **length** of the path is the number of edges in it.

Thus { (v,w), (w,j), (j,z), (z,q) } is a path from v to q of length 4. Here is a planar representation of this graph:

(1)



A graph is said to be **connected** if for any two vertices in V there is a path from one to the other. Graph (1) above is an example of a connected graph. If (w,j) was missing from (1), then it would no longer be connected. Note that by this definition a single vertex is considered a connected graph.

Consider a graph G with a vertex set V and an edge set E. A **subgraph** of G is a graph H with a vertex set contained in V and an edge set contained in E. If the edge set of H consists of all edges of G whose endpoints are in the vertex set of H, then H is said to be an **induced subgraph** of G.

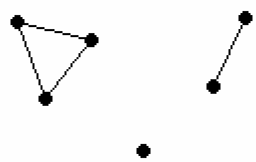
Thus, the edge (v,w) and the vertices {v, w, j} form a subgraph of (1). It is not an induced subgraph, since the edge (w,j) is in (1) and although both its vertices are in the subgraph, it is not an edge of the subgraph.

A **partition of a set** is a collection of disjoint subsets (called blocks) such that the union of all the blocks is the whole set. One place partitions are used is in the study of Riemann integration as means for dividing up the real number line. A partition of the interval (1,10), for example, could be the intervals (1,2], (2,3], … , (9,10).

A **partition of a graph** G is a partition of both its edges E and its vertices V into subsets $\{V_j\}$ and $\{E_j\}$ such that there exists a graph G_j with V_j as its vertex set and E_j as its edge set.

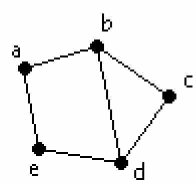
A graph can be partitioned into its **maximal connected subgraphs**, which are called its **connected components,** if there are no edges that go between the subgraphs (since otherwise these edges will be lost in the partitioning).[1] This definition derives from the fact that if a graph is not connected then it can be partitioned into subgraphs, each of which is connected, and none of which are connected to each other. For example, graph (1) has 1 connected component. Graph (2) below has 3 connected components.

(2)



A **cycle** in a graph is a path that starts at the same vertex at which it ends. A **chord** of a cycle is an edge among its vertices that is not part of the cycle. In graph (3) below, the edges { (a,b), (b,c), (c,d), (d,e), (e,f) } form a cycle and the edge (b,d) is a chord.

(3)



There is a standard notation for several kinds of graphs that are of general interest.

The graph $\mathbf{C_k}$ is a cycle of length *k*, consisting of *k* vertices and *k* edges.

[1] We can of course, if we want to, define partitions of the edges set of a graph, and let the vertex sets of the resulting graphs overlap.

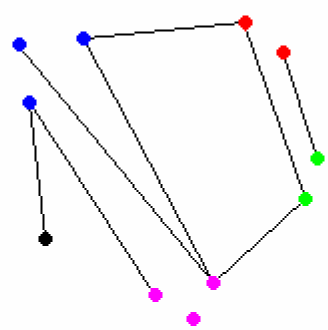
A **complete graph** with n vertices, written K_n, is a graph with *n* vertices that contains every possible edge. The number of edges in K_n is the binomial coefficient $\binom{n}{2}$ (see the exercises for more info).

14.2 Coloring Graphs

One concept that is helpful for characterizing graphs is that of graph coloring. What we do is assign colors to each of the vertices in a graph with the condition that no two vertices that share an edge can have the same color. There are many ways to assign colors to a particular graph, some of which require more colors than others. Of course, any graph that has at least one edge will require at least two colors, and many will require more. The minimum number of colors required to color a graph is called the **coloring number** of the graph. A graph that can be colored with at least *k* colors is said to be ***k*-colorable**.

A graph whose vertices can be partitioned into *k* subsets such that no vertices that share an edge are in the same subset is said to be ***k*-partite**. Thus, a bipartite graph is one whose vertex set V can be split into two parts, and all edges contain one vertex from each part. Here is an example of a 5-partite graph with each of the 5 subsets colored to make them easier to distinguish:

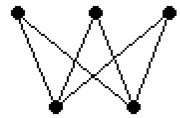
(4)



We can see from the coloring of graph (4) above that it is 5-colorable in addition to being 5-partite. In fact, from the definition of a *k*-partite graph, we can see that any *k*-partite graph will be *k*-colorable (see the exercises for more info).

$\mathbf{K_{n,m}}$ is the notation we use for a complete bipartite graph between vertex sets of size n and m. Thus, it consists of two sets of vertices (one of size *m* and one of size *n*) and all possible edges with one vertex in set *m*, the other in set *n*, and no edges within each of these two sets. Here is an example of a $K_{3,2}$ graph:

(5)



A basic question is: *under what circumstances is a graph bipartite (that is, two-colorable)?*

We can give the following necessary and sufficient condition for bipartness or two-colorability:

A graph will be two colorable if it has no odd length cycles.
If a graph has an odd length cycle, then it cannot be two colorable.

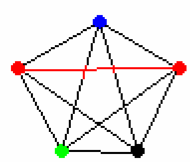
We will prove the second part first. Let G be a 2-colorable graph. Suppose, in order to reach a contradiction, that G has an odd length cycle. We will try to color the cycle one vertex at a time. We start at vertex v_1 and give it color A. This means that all of v_1's neighbors (the vertices with which it shares an edge) must have color B, since no two vertices sharing an edge can have the same color. So v_2 has color B, v_3 has color A, and so on. We see that all the odd vertices in the cycle will have color A and the evens will have color B. Since the cycle is of odd length, it has an even number of vertices, so the vertex right before v_1 will be an odd vertex and thus be given color A. But this is the same color as v_1 and since they share an edge, this is a contradiction. Thus, the graph must not be 2-colorable.

We can use a similar argument to prove that any graph that has no odd length cycle is bipartite (see exercises for more info).

□

In a similar vein, it is not possible to color the complete graph K_n with fewer than n colors. In any coloring with fewer colors, two vertices must have the same color, but since a complete graph has an edge between every pair of vertices, this violates the condition that all edges must contain vertices of different colors. Below is an unsuccessful attempt to 4-color K_5 (note the red edge between the two red vertices indicating a violation of the coloring rules):

(6)



14.3 Perfect Graphs

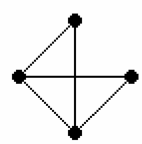
A complete graph is often called a **clique**. The size of the largest clique that can be made up of edges and vertices of G is called the **clique number** of G. Recall from the previous section that any clique of n vertices must be n-colorable. This tells us that:

$$\text{Coloring number of G} \geq \text{Clique number of G}$$

The coloring number of a graph can be strictly greater than its clique number, as we have already seen for odd cycles of length 2 or more. These have clique number 2 (which means it contains no triangle) but coloring number 3 (since we proved that odd cycles are not 2-colorable).

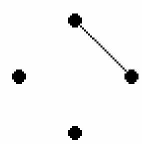
If the coloring number and the clique number are the same for every induced subgraph of G, we call G a **perfect graph**. Here is an example of a perfect graph. Look at all the induced subgraphs and prove to yourself that this is in fact a perfect graph:

(7)



The **complement** of a graph G is the graph on the same vertex set V, whose edges are precisely those that are not in the edge set of G. Thus, the edge set of G and of its complement include all the edges of the complete graph on V, and the edges of G and its complement do not overlap at all. As an example, below is the complement of (7):

(8)



A famous result of graph theory is **The Perfect Graph Theorem**, which reads:

A graph is perfect if and only if its complement is perfect.[2]

The coloring number and clique number of the complement of G are parameters of interest by themselves. The complement of G has all possible edges not in G. Thus, a clique in the complement of G is a set of vertices among which there are no edges of G. We call this an **independent set** of G; a set of vertices unrelated by any edge of G. For example, the two vertices in the upper right of (7) are an independent set because there are no edges connecting them.

The number of vertices in the largest possible independent set of G is called the i**ndependence number of G.** Thus, from the previous paragraph we can see that,

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[2] We could just as easily say, "If a graph is perfect then its complement is perfect", since the complement of the complement of G is G.

Clique number of the complement of G = Independence number of G

In these terms, we can describe the coloring number of G as the smallest number *k* such that we can partition the vertices of G into *k* independent sets. In other words,

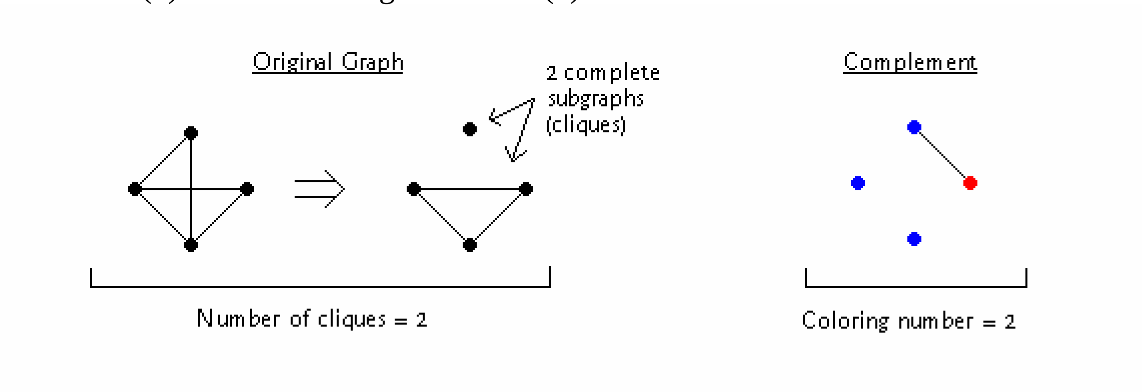
Coloring number of G = smallest # of independent sets of G

This follows from the fact that any two vertices of the same color cannot share an edge, and thus the vertices of each color form an independent set. For an example, look back at (4) and see how each color is an independent set.

We can similarly see that the coloring number of the complement of G is the smallest number k' such that we can partition the vertices of G into k' cliques. In other words,

Coloring number of the complement of G = smallest # of cliques in G

This follows from the previous statement, along with the fact that an independent set in the complement of G is a clique in G. As an illustration, we will look at the clique number of (7) and the coloring number of (8) to see that this holds:



Using these results, we can now rephrase the perfect graph theorem (version II):

If for any induced subgraph H of G, we can partition the vertices of H into a number of cliques given by the size of H's largest independent set, then we can partition G (or any of its induced subgraphs) into a number of independent sets given by the size of its largest clique

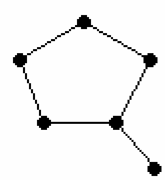
To see how this matches the original statement of the theorem, think about what it means for G and its complement to be perfect, and use the boxed equalities and definitions above.

As an illustration, we will show how "*For any induced subgraph H of G, we can partition the vertices of H into a number of cliques given by the size of H's largest independent set*" is equivalent to saying that the complement of G is perfect: This statement tells us that the smallest # of cliques of H $\geq$ Independence number of H which means that [the smallest # of cliques of H] = [Independence number of H], and we know from above that [the smallest # of cliques of H] = [Coloring number of the complement of H] and that [the independence number of H] = [Clique number of the complement of H]. Putting these equalities together we get that [the clique number of the complement of H = Coloring number the complement of H], and thus the complement of H is perfect. We can use a similar argument for the second half of the above restatement of the perfect graph theorem (see exercises for more info).

Note that in any partition of V into cliques, since each vertex of an independent set, I, must end up in a clique that contains no other member of I, the total number of blocks of the partition must be at least the maximum size of any I (which is the independence number). The same statement holds with the words "clique" and "independent set" reversed.

This tells us that the concept of perfect graphs, and the restatement of the perfect graph theorem, requires that you can partition the vertices of any induced subgraph of G into a number of cliques given by the independence number of that subgraph. Thus, the reason that our restatement of the perfect graph theorem involves induced subgraphs is that it would be false if we were to apply its condition to G but not to its induced subgraphs. We can see this by considering the following graph, on 6 vertices, that can be described as a 5-cycle with another edge linking the sixth vertex to one vertex of the cycle:

(9)



For this graph, the independence number is 3 and it can be partitioned into three cliques. On the other hand, the clique number is 2 and it cannot be partitioned into two independent sets (verify these assertions yourself).

However, the induced subgraph on the five vertices that form the cycle has independence number 2 and clique number 2 and can only be partitioned into 3 cliques and 3 independent sets. Thus, by the perfect graph theorem, neither Q nor its complement are perfect.

This leads us to a third way to state the perfect graph theorem (version III):

If you cannot partition the vertices of G into a number of cliques given by the size of its largest independent set, then G has an induced subgraph H that cannot be partitioned into a number of independent sets given by H's clique number.

Try to figure out yourself how this is equivalent to the first two versions of the theorem.

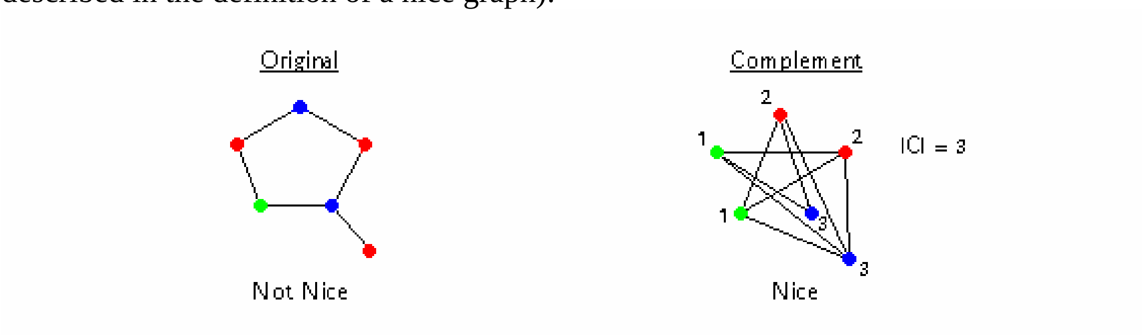
14.4 Nice Graphs

We will now look at another way to describe graphs.

A graph G is a **nice graph** if given its maximum size clique C, whose size we shall denote as $|C|$, we can assign integers 1 to $|C|$ to the vertices in C and can extend that assignment to all the vertices in V so that for each j, the vertices assigned the letter j form an independent set. This is really saying that if G is nice then we can partition G into a number of independent sets equal to the clique number of G, which, if you did exercise 4, you will see is the statement that G is nice when its clique number and coloring number are the same.[3]

A graph is **c-nice** if its complement is nice, which means that its independence number and the smallest number of cliques that its vertices can be partitioned into are the same (this follows from the results of the previous section).

A graph is **very nice** if both G and its complement are nice (that is, G is both nice and c-nice). In this terminology, graph (9) above is c-nice, but not nice, and therefore not very nice. (9) is clearly not nice because it has an odd cycle and thus must be at least 3-colorable, but its clique number is 2. The complement of (9), on the other hand, has clique number 3 and has coloring number 3. To illustrate this, we have drawn (9) and its complement below (since the complement is nice, we have added the numbering as described in the definition of a nice graph):



A **minimally not nice graph** is a graph that is not nice, but all its induced subgraphs are nice. Similarly, a **minimally not very nice graph** is a graph that is not very nice, but all its induced subgraphs are very nice. A graph that is **not nice at all** is graph where neither it nor its complement are nice.

If a graph is minimally not very nice, then either G or its complement are not nice, but the induced subgraphs are all very nice. This means that if either G or its complement is nice, then that graph is nice, and all its induced subgraphs are nice, and thus it is perfect. However, looking at the perfect graph theorem, this would imply that both graphs are perfect, which contradicts the fact that it is not very nice.

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[3] Note how this differs from the definition of a perfect graph, which stipulates that all induced subgraphs must also have their coloring numbers equal to their clique number. Nevertheless, we can see that although not all nice graphs are perfect, all perfect graphs are nice.

The logic employed above allows us to phrase the perfect graph theorem in terms of niceness (version IV):

Every minimally not very nice graph is not nice at all

There is a stronger statement that had been a conjecture for about 40 years (called the Berge conjecture) but has just recently been proven. It is called the **Strong Perfect Graph Theorem.** One standard form of the theorem is (version I):

A graph is perfect if and only if neither the graph nor its complement contains an odd graph cycle of length at least 5 as an induced subgraph.

We can also phrase this theorem in terms of niceness (version II):

The only minimally not very nice graphs are odd cycles of length 5 or more without chords, or the complements of these.

It is easy to prove that these chordless cycles of length 5 or more are not nice at all (see the exercises for more info). Thus, we see that the perfect graph theorem (version IV) is an immediate consequence of the strong perfect graph theorem (version II). Another consequence of the strong perfect graph theorem is that a graph will be very nice unless it or its complement contains a chordless odd cycle.

14.5 Properties and Applications of Nice Graphs

Nice graphs have some occasionally useful properties. One is as follows:

The size of the vertex set of a very nice graph G is at most the product of its clique number and its independence number.

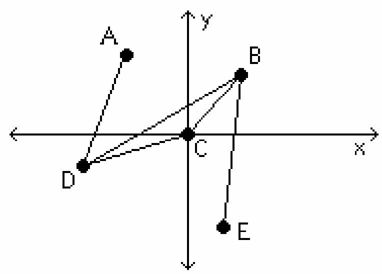
This statement follows immediately from these two facts:

1. If G is very nice then we can assign an ordered pair (j,k) to each of its vertices such that those vertices with the same first component form an independent set and those with the same second component form a clique. Also, j will run from 1 to |C|, the clique number, and k will run from 1 to I, the independence number.

2. No two vertices can have the same assigned pair; if they form an edge of G they cannot be in the same independent set, and if they do not form an edge they cannot be in the same clique

The same idea used in this last result can be applied to an arbitrary set of N distinct points in the plane, each described by coordinates (j,k). We can ask: *what can we* *say about the size* (|I| or |D| respectively) *of the largest monotonic increasing or decreasing subset of these N points?* [4]

We can define a graph whose edges are the pairs such that the line between them has non-negative slope. A monotonic increasing set will be a clique in this graph, and a decreasing one will be an independent set. We have an example of such a graph below. We see, for example, that points D, C, and B form a monotonic increasing set (a clique) and points A, C, and E form a monotonic decreasing set (an independent set):



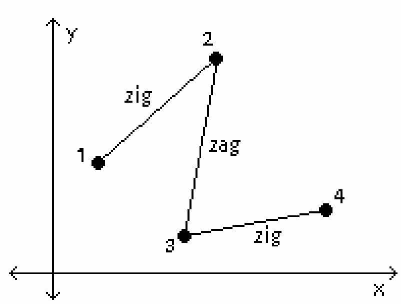
We want to show that:

N is at most |I||D|

To prove this statement, using the first result of this section and the strong perfect graph theorem, we need only show that the graph here defined can contain no chordless odd cycle of length 5 or greater, and thus is very nice (the same result will hold by symmetry for its complement).

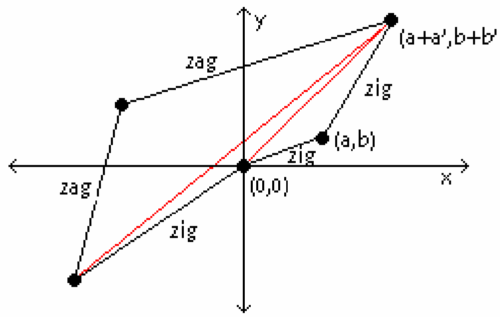
We will show that the greatest chordless odd cycle that can exist is a triangle. Suppose we have an odd cycle in a graph of the type described above. Let us call one point (0,0) and suppose that the next point is (a,b) with positive a. We can deduce that b is positive as well, since the slope is positive.

We will arbitrarily assign a direction to the cycle, so we can talk about vertices that come after others. Let us call an edge in which one vertex has larger coordinates than the vertex before it a "zig" and one in which coordinates go down a "zag". Note that both zigs and zags have positive slope:



[4] A monotonic increasing set is a set of elements $\{x_{1,}, x_2, \dots, x_n\}$ where $x_1 \le x_2 \le \dots \le x_n$. That is, every element in the set is greater than or equal to the element than comes before it. A monotonic decreasing set is the same, except each element is less than or equal to the element that comes before it.

Our cycle will be composed of these zigs and zags. Since it is an odd cycle, there must be two adjacent zigs or two adjacent zags at least. Suppose then that there are two adjacent zigs, and let the starting vertex of the first one be (0,0). The next vertex around the cycle will then be (a,b) and the next (a+a',b+b') with a, a', b, and b' all positive. This implies that there is an edge from (0,0) to (a+a',b+b') which makes our cycle a triangle or makes this edge a chord. Thus, we conclude that the only chordless odd cycles in our graph are triangles. We provide an illustration of an example odd cycle to show how consecutive zigs and consecutive zags can create chords (here in red):



Therefore, using the argument from above, we get that N ≤ |I||D|.

□

A common application of this statement is that:

A permutation of N integers contains either an increasing or decreasing sub-permutation of length at least $\sqrt{N}$.

14.6 Planarity of Graphs

**Editor's Note: In some of the arguments in this section, we shall make some intuitive assumptions about the properties of curves on a plane. Many of these assumptions are in fact based on topological concepts, and could be rigorously defined if necessary.*

A graph so far is an abstract thing, a creation of your mind. It consists of a set of vertices and of edges.

We can, given a graph, attempt to draw it on a piece of paper, representing the vertices by points and its edges by either straight lines or curves.[5]

We then define a graph G to be **planar** if it can be so drawn without any of the curves or lines representing its edges crossing one another or meeting any other vertex on the way from one vertex to the other. Note that there might be many ways to draw the graph such that its edges cross, but as long as there is one way to draw it such that no edges cross, then it is planar. For example, looking at graph (7) you may not think that it

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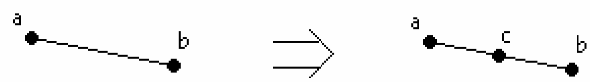
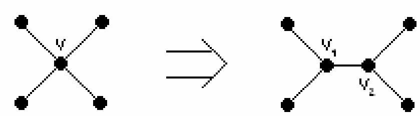
[5] In the previous sections, we have been drawing many graphs in order to illustrate concepts, but keep in mind that a graph is really an abstract concept that does not have to have a physical representation.

is planar because two of its edges cross. However, the same graph can be drawn in a different way such that no edges cross and we see that it is in fact planar:

It will be useful later to have a notion of **adding a vertex to an edge** and **splitting a vertex.** Adding a vertex in the middle of an edge here means replacing an edge (a,b) by two new edges (a,c) and (c,b):



When we split a vertex, we take a vertex *v* and replace it by 2 vertices v_1 and v_2 so that each edge containing v contains one of these, and there is an additional edge containing v_1 and v_2. For example:



We will say that a **subdivision** of G is any graph that is obtainable by adding vertices to an edge of G or splitting vertices of G. (For the sake of completeness, we will consider G to be a subdivision of itself).

We define the **degree of a vertex** to be the number of edges that contain it as an endpoint (we will use this definition later in the section).

One big question we would like to answer is: *is there a convenient way to characterize planar graphs?*

Before answering this question, we make some remarks, which we will prove

1. There are two fairly small graphs that are not planar, K_5 and $K_{3,3}$
2. We can add vertices in the middle of any edge of a non-planar graph as we like and that will not help to make it planar.
3. We can split any vertex apart and that will not make a non-planar graph planar.
4. No graph that contains K5 or K3, 3 or something obtained from these by adding vertices in the middle of edges or splitting vertices can be planar.

To prove 1, we will use the following lemma:

If we take two different drawings of either K_5 (or more generally K_{2j+1}) or K_3 (more generally $K_{2j+1,2j+1}$) in the plane, then number of crossings between edges <u>*whose vertex*</u>

<u>sets are disjoint</u>[6] *has the same value mod 2 in each drawing (we count a point of tangency between two as edges as either 2 or 0 crossings).*

It follows immediately from this statement that if we can find a drawing of either K_5 or $K_{3,3}$ with an odd number of crossings, it cannot be drawn with no crossings. (see exercises for more info).

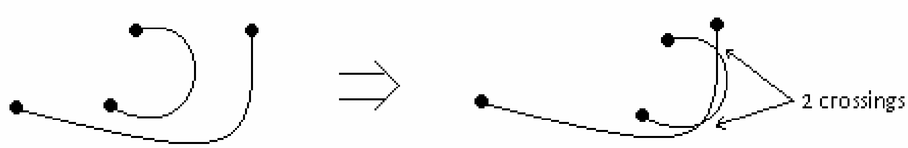
So let us prove the lemma:

We start with two different drawings of the same graph, with vertex sets the same for each. We will take an edge of the first graph and move it slowly and continuously until it reaches the position of the same edge in the second drawing. We will do this for each edge of the first graph. When we are done, the two drawings will have the same number of crossings between edges with disjoint endpoints, since they will become identical.

To prove this result, we look at all the possible ways that the number of crossings in graph one could change, and show that the total number of crossings always remains the same, mod 2.

How could it change? If the edge *m* being moved does not either become tangent to another edge *q* or cross over one the endpoints of *q*, the number of crossings between *m* and *q* will not change in any way. The crossings, if any, will merely slide along *q*.

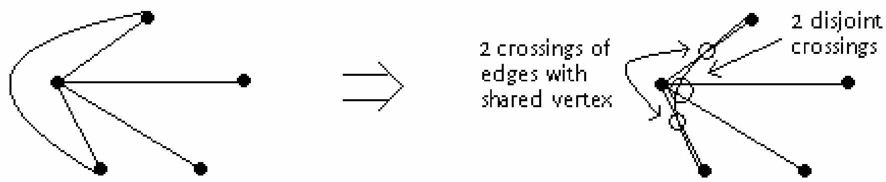
When *m* and *q* become tangent and then cross, or become tangent and uncross, the number of crossings between *m* and *q* will change by 2, which mod 2 is 0:



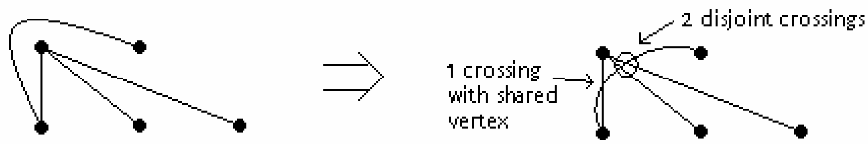
When *m* crosses over an endpoint *v*, the number of crossings of *m* with every edge of containing *v* as an endpoint will change by 1, either up or down.

In the case of K_{2j+1}, since every vertex shares an edge with every other vertex, two of these crossings will involve edges that share endpoints with the two ends of *m*. The number of crossings not including these two (since these edges do not have disjoint vertex sets) will therefore change by an even number when *m* passes over *v* (since the vertices of K_{2j+1} have even degree), which mod 2 is 0.

[6] The reason that it must be crossings with disjoint endpoints is because if two edges share an endpoint then we could make them cross each other an arbitrary number of times if we wanted to, but it is easy to see that we could always draw them in such a way that they do not cross at all.



In the case of $K_{2j+1,2j+1}$, when m crosses over *v*, exactly one of the edges coming out of *v* will share an endpoint with *m*, and since the degree of $K_{2j+1,2j+1}$ is odd, the number of crossings between edges which do not share an endpoint will change by 0, mod 2, when *m* passes over *v*.



We conclude that the number of crossings in either case can never change mod 2 as the first drawing is transformed into the second one. Thus, the number of crossings mod 2 must have been the same to begin with, which is what we set out to prove.

□

We prove remark 3 above by noticing that if a planar graph has a split vertex *v*, we can transform its drawing so as to make the edge between v_1 and v_2 shorter and shorter, until it entirely disappears, without introducing any crossing of edges. In the process, we undo the vertex splitting.

Remarks 2 and 4 do not require proof, as 2 is apparent and 4 follows directly from remark 1.

We turn then to the question posed above of how to characterize planar graphs. It turns out that there is a very nice theorem, called **Kuratowski's Theorem**, which tells us when a graph is non-planar:

A graph is planar if it does not contain a subgraph that is a subdivision of K_5 *or* $K_{3,3}$

This theorem tells us that the absence of these two configurations or their subdivisions, which we have seen is enough to ruin planarity, is enough to ensure planarity.

We will now give an outline of the proof of Kuratowski's theorem.

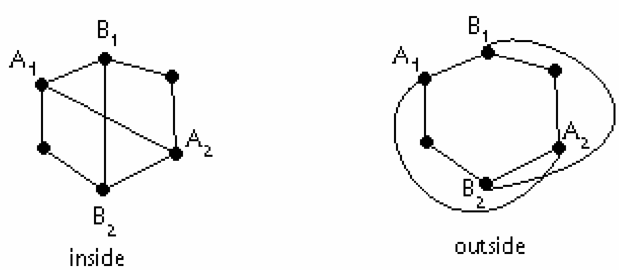
Suppose G is a minimal non-planar graph. This is a graph such that if you remove one edge then it is planar (we can start with any non-planar graph and remove irrelevant edges until it is minimal in this sense).

Our plan is to find a simple cycle C in G such that the graph G – C has at least two connected components. (G – C is the graph obtained from G by omitting all edges and vertices of C, except for vertices of C that also have edges that are not in C). We say that two edges or vertices in G – C are in different connected components if you any path between them has to go through a vertex of C.

To find C, we remove an edge E from G such that G – E is planar and draw it in the plane. We then find a path in G from one vertex of E to the other that crosses as few boundaries as possible (you have to cross some or G is planar). This will be a path that goes through G but goes through no vertex of G except those of E. We now look at the outer boundary of the union of the faces of G – E that this path goes through. This is a cycle, and E is a chord of the cycle. If G is non-planar, then this cycle will fit the description of C given above.

The rest of the graph not in C can be divided into several "bridges". A **bridge** is one of the connected components of G – C. The simplest example of a bridge is a chord of the cycle C. In fact, it turns out that if G is a minimal non-planar graph, then all of its bridges are chords of C (see exercises for more info).

We say that two chordal bridges A (with vertices A_1 and A_2) and B (with vertices B_1 and B_2) are "incompatible" if their vertices, moving around C, appear as A_1, B_1, A_2, B_2. This means that two incompatible bridges cannot both be drawn inside (or outside) the cycle without crossing:

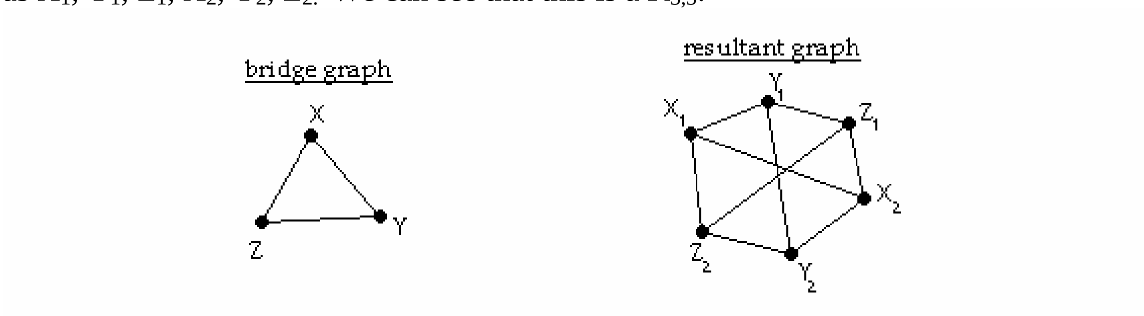


We next define a **bridge graph**. Its vertices are the bridges, and there is an edge connecting any pair of bridges that are incompatible.

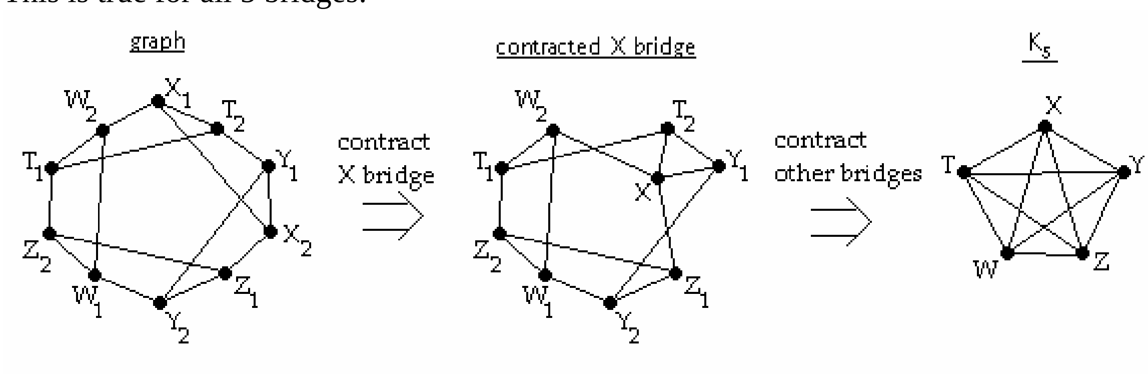
If we take a pair of incompatible bridges, we can draw one of them inside the cycle and the other outside the cycle without any crossings. If we could do this for every pair of incompatible bridges, then we could draw G with no crossings and it would be planar. This is equivalent to saying that if the bridge graph can be divided into two groups of vertices such that there is no edge between any of the vertices in a group, then G is planar. Recall that a graph which can be divided into two groups in this way is said to be bipartite. Therefore, since G is non-planar, its bridge graph is not bipartite.

We saw in section 14.2 that if a graph is not bipartite then it contains an odd cycle. This means that the bridge graph of G must contain an odd cycle. Thus, we can prove Kuratowski's theorem if we can show that any odd cycle in the bridge graph requires a configuration in G that is obtained by vertex splitting or edge subdivision from K_5 or $K_{3,3}$.

Suppose, for example, that the odd cycle of the bridge graph is a triangle. Then we have three bridges in G: X, Y and Z. The pair X Y is incompatible, as is Y Z and X Z. This means that the endpoints of the three bridges must be arranged in order around C as X_1, Y_1, Z_1, X_2, Y_2, Z_2. We can see that this is a $K_{3,3}$:



If we have a 5-cycle, then the endpoints of the five bridges will be arranged in order around C as X_1, T_2, Y_1, X_2, Z_1, Y_2, W_1, Z_2, T_1, W_2. We can then construct a K_5 by contracting each bridge to a single vertex. Notice that X_1 is adjacent to T_1 and W_2 on C and X_2 is adjacent to Y_1 and Z_1, so when contracted X will be adjacent to T, W, Y, and Z. This is true for all 5 bridges:



Larger odd cycles in bridge graphs can be contracted into K_5 graphs by contracting three of the chords to make three single vertices and judiciously contracting all the other vertices into the last two vertices of the K_5. We will leave the 7-cycle bridge graph as an exercise (see exercises for more info).

This shows that every non-planar graph has a K_5 or $K_{3,3}$ as its subgraph, and proves the theorem.

□

Exercises

Exercise 1 Show that the number of edges in a complete graph with n vertices, K_n, is given by $\binom{n}{2}$, and that this is equal to $\frac{n(n-1)}{2}$.

Exercise 2 Show that any k-partite graph is k-colorable.

Exercise 3 Prove that any graph that has no odd length cycle is bipartite (2-colorable).

Hint: You can use a similar argument to the one we used to prove that a graph that has an odd length cycle is bipartite, but in this case we get a coloring instead of a contradiction. We can start anywhere on the graph by coloring one vertex v color A, v's neighbors color B, their neighbors color A, and so on until every vertex that can be reached by a path from v is colored. The absence of odd cycles means that each vertex will have the same color every time it is colored.

Exercise 4 Show that the statement "*we can partition G (or any of its induced subgraphs) into a number of independent sets given by the size of its largest clique*" (from version II of the perfect graph theorem) is equivalent to saying that G is perfect.

Exercise 5 Show that chordless cycles of length 5 or more are not nice at all.

Exercise 6 Show that if G is a minimal non-planar graph, then all of its bridges are chords of C

Exercise 7 If G has a 7-cycle bridge graph then show how G can be contracted into a K_5 graph.

Additional Sources

Mathworld-A Wolfram Web Resource. http://mathworld.wolfram.com

~Edited by Jacob Green