18. Counting Patterns

18.1 The Problem of Counting Patterns

For this discussion, consider a collection of objects and a group of permutation symmetries (G) that can act on the objects. An object is not a mathematical term, we simply mean that the permutation symmetries are acting on any type of object; in the next couple pages we will refer to an example that uses a beaded necklace as the object in question. Also, we mean that each element g of G can be applied to any given object A to form a new object. In some cases the new object is different from A, but in some cases A is left unchanged, such that gA = A.

We want to determine the number of distinct patterns of our objects there are. Two objects A and B are indistinct patterns if the application of one of the symmetries g to the object A brings it into B, such that gA = B.

If you are confused about the two scenarios just mentioned (gA = B and gA = A), consider this example. The object in question is a necklace, and the group elements are things you can do to reorient the necklace. The necklace is made of colored beads and we clone it such that we have two different necklaces that have the same pattern of beads on them. Then, we place the two necklaces one on top of the other such that the colors line up. If we rotate one, the two necklaces still have the same pattern, but look slightly different. This is an example of two objects having the same pattern such that gA = B. In other words, the symmetry g acting on A is a certain rotation such that A becomes B. On the other hand, if you forget about B and just take A and rotate it 360^o we still have A. This is an example of when gA = A. g in this case is the 360^o rotation. More about necklaces later.

A **stabilizer** of A, denoted as S(A), is the subgroup of G such that all g in the subgroup leave A unchanged when applied to it (gA = A). If g is not in S(A), gA is another object with same pattern as A. The set of these objects with the same pattern as A is called the **orbit** of A under action of G.

Note that the **cardinality** (size, number of elements) of S(A) is the same as the cardinality of the stabilizer of every element in the orbit of A. To rationalize this, let A and B be objects, where B is the member of A's orbit with the largest stabilizer. Let g be some symmetry such that gB = A.

We know S(B)B = B, where S(B) is any element in B's stabilizer, since this is the definition of a stabilizer. Thus gS(B)B = A. Since B = g^{-1}A, gS(B)g^{-1}A = A. Thus, by definition, gS(B)g^{-1} is in the stabilizer of A, for any B in A's orbit. S(A) then has the same cardinality of S(B).

Also, since gS(B)B = A, the coset of S(B) in G given by gS(B) defines a set of group elements that take B into A. The cardinality of gS(B) is the same as S(B) of course, written |S(B)|. Thus, there are |S(B)| elements that take B into A. This is true for any A in the orbit of B.

Furthermore, we know that the size of the orbit of B must be the number of cosets of S(B) in G. Since |S(B)| = |S(A)|, we can conclude that the size of the orbit of A is the number of cosets of S(B) in G also.

From Lagrange's theorem it follows that:

[Size of the orbit of B] * |S(B)| = |G|

That is, the size of the orbit of B is the order of G divided by the order of B's stabilizer.

Now we realize that what we originally referred to as a unique pattern is simply an orbit under the action of G. So in order to count patterns, we need to count the number of orbits of our objects under the action of G. We will denote the orbit of A under G as O(A).

To determine the number of orbits, we can sum up the total weight of all the objects in the collection, where the weight of each object in orbit A is $\frac{1}{|O(A)|}$. This sum is the number of orbits or patterns.

We know that |O(A)| is equal to the order of G divided by the order of the stabilizer of B, using the formula above. We also know that the sum of |S(A)| for all A is the same as the number of pairs consisting of an object and a group in which the group element stabilizes the object. Another way to look at this sum is the sum over all g in G of the number of objects stabilized by g.

Thus, *the number of patterns/orbits of a set of objects under the action of a group G is the sum over all group elements g of the number of objects stabilized by g, divided by the order of G.*

Note that this number is also the average number of objects stabilized by a group element g of G. Also note that a function defined on our objects that takes the same value for each member of a conjugate pair will be constant on any orbit.

Based on these two facts, we can conclude that *the sum of f over all patterns is equivalent to the average over all group elements g of the sum of f over the patterns stabilized by g, given that f is constant on the orbit*. The proof of this is left as an exercise.

As a consequence of this result, the average of the number of patterns or the sum of f over patterns stabilized by g will be the same for all g in the same conjugacy class. Thus, the number of terms in the computation of the average number of patterns is at most the number of conjugacy classes in G. As we have seen, for S_9 there are only 30 partitions of n, and the number of conjugacy classes in the this group is the number of partitions, so that you could compute this sum of S_9 by hand if you had to, though the size of the group is in the hundreds of thousands.

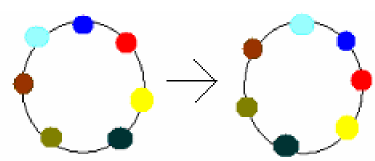
18.2 Counting necklaces of n colored beads

One application of these principles of counting patterns is the creation of n-bead necklaces, each of which is one of k distinct colors. The question is: how many distinct necklaces can we create?

To clarify, two necklaces are distinct if there is no way that they can be placed one on top of another such that the colors of the beads all line up.

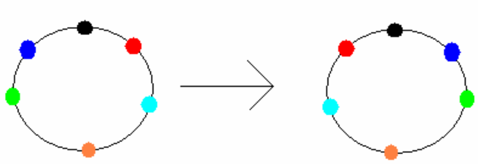
The set of operations that transform two identical necklaces from different orientations into necklaces that are lined up exactly as mentioned composes a permutation group of symmetries, which we will call G_n.

If we number the beads in a necklace (1,2,3…n) we can perform a **cyclic permutation**. This means moving the necklace such that bead 2 is now in the position where bead 3 was, bead 3 is in position 4, and so on so that the general structure is (n, 2, 3, 4,…1). Naturally, we call this a rotation.

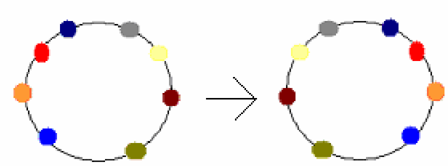


Example of a rotation

Another permutation is a reflection. If n is odd, a reflection entails keeping one bead fixed, while swapping each bead of distance j from the fixed bead with the corresponding bead of distance j in the other direction from the fixed bead. If n is even, there are two ways to perform a reflection. One is holding a point between two beads constant, and swapping all pairs of beads with equal distances from the point. The other method is to hold two beads on opposite sides of the necklace constant, and swap corresponding pairs. See the diagram for clarifications.



Reflection of necklace having even n, holding two opposite beads fixed



Reflection of necklace having even n, holding point between two beads constant

Let's look at the cycle structure of these permutation symmetries. For reflections, if n is odd, we have one block of size 1, and (n-1)/2 of size 2. There are n possible reflections, since we could fix any one of the n beads. For an even value of n with the two fixed beads, we have 2 blocks of size 1 (for the fixed beads) and (n-2)/2 blocks of size two. There are n/2 such reflections. For an even value of n with no fixed beads there are n/2 blocks of size two, and there are n/2 such reflections.

If we operate about the necklace without changing anything (360^o rotation, for example), we call it the identity, which has a cycle structure of n blocks of size 1. As for rotations, if n is prime, there is only one possible cycle structure having 1 block of n. If n is not prime, then there are n/k cycles of size k for each divisor k of n. That is, if n is 6 for example, cycle structures possible include 1 block of 6, 2 blocks of 3, or 3 blocks of 2. In any case, there are only n-1 total possible rotations. Thus, the group of symmetries Gn has 2n elements.

Consider the example where n = 10. The following congruence classes of elements exist:

Type	Quantity	Cycle Structure
Identity	1	1_{10}
Reflection	5	2_5
Reflection	5	$1_2 2_4$
Rotations	4 (rotate by 2, 4, 6, 8)	5_2
Rotations	4 (rotate by 1, 3, 7, 9)	10_1
Rotation	1 (rotate by 5)	2_5

Note on cycle structure notation: Base number is cycle size, subscript is number of such blocks.

Since the formula for number of patterns of coloring gets contributions from each of the distinct cycle structures, there will be five terms since there are five types of cycle structure. Since there are 20 total elements (1 + 5 + 5 + 4 + 1), the size of the orbit is 20, and thus the weights of each term will be the number of elements in each class divided by 20. The weights are: 1/20 for the identity, 4/20 for one 10 cycle, 4/20 for two 5 cycles, 5/20 for 5 reflections with structure $2_4 1_2$, and 6/20 for 5 rotations and a reflection with structure 2_5.

Now that we have the weights for each term, we just need to find out how many colorings are stabilized by each of these. We will do this in a counterintuitive way by introducing some notation that will make things more complicated, but will allow us to solve some more difficult problems later on.

We start by introducing the variable t_s for each color s. t_s is one when color s is available to use in the necklace, 0 otherwise. Realize that in each cycle, the same color must be used, in order for the symmetry to be stabilizing. That is, in a cycle of 5, all five beads must be the same color. Realize also that if the action of the symmetry on the necklace breaks the necklace into q cycles (for example, q=2 cycles of 5), and there are C total colors to be used, then there are C^q possible colorings, using simple combinatorics. The number C can be replaced by the sum over all s of t_s^k, where k is the size of the cycles. Thus, in our notation, the number of objects stabilized by a certain group element is: $(\sum_s t_s^k)^q$.

If we incorporate the weights of each class of cycle structures and sum over all of them, we get the following way to count patterns in our 10-bead example:

$$\frac{1}{20}(\sum_s t_s^{\ 1})^{10} + \frac{4}{20}(\sum_s t_s^{\ 10})^{1} + \frac{5}{20}(\sum_s t_s^{\ 5})^{2} + \frac{5}{20}(\sum_s t_s^{\ 2})^{4}\ (\sum_s t_s\ )^{2} + \frac{6}{20}(\sum_s t_s^{\ 2})^{5}$$

If there are seven colors to choose from, and we have an unlimited number of each bead, we can set $t_s = 1$ for s = 1…7. Then we get: $\frac{7^{10}}{20} + \frac{7}{5} + \frac{7^2}{4} + \frac{7^6}{4} + 3 * \frac{7^5}{10}$

You may now be wondering why we introduced such complex notation to solve a problem that could have been solved with elementary combinatorics. The reason is that using this notation, we can solve many more similar problems with slight adjustments. For example, how many colorings are there with exactly 3 red beads, and 7 non-red beads, all other parameters being the same.

If there are three red beads, t_{red} should occur with an exponent of 3 (t_{red} is the t_s that corresponds to red), using our notation. We can solve this by finding the coefficient of t_{red}^3. This could only occur in the identity term and the second to last term. Thus, we set $t_s = 1$ for all of the other six colors and find the coefficient of t_{red}^3. Using binomial expansion, this comes out to $\frac{1}{20}$ C(10,3)6^7 + $\frac{1}{4}C(2,1) * 6 * C(4,1) * 6^3$. This simplifies to $6^8 + 2*6^4$.

18.3 Applications to counting tree patterns

We can also count the number of patterns of trees on n vertices this way. Although our group here is the symmetric group on n symbols and this group grows rapidly with n, the number of congruence classes grows fairly slowly.

To apply this you have to be able to count how many trees will be stabilized by each group element.

Again the identity will stabilize them all, and for each other congruence class you must count how large it is, divide that by the order of S_n and multiply by the number of trees stabilized by group elements in that class.

Typically you can choose any tree on the fixed points of the congruence class, then add on a leaf for each cycle to it. When cycles have the same length you get trees when one is added to the other (in any order), but when cycles have different lengths their vertices cannot be added to one another so as to maintain symmetry.

Thus, for example the cycle structure $1^3 2\ 3$ on 8 vertices will stabilize trees that starting with 3 vertices have a pair of leaves added to any one of these and a triad of leaves added to another. There are 3^1*3*3 such trees stabilized by each such cycle structure, and there are C(8;3,3,2) such structures where this C(8;3,3,2) is the multinomial coefficient 8!/(3!3!2!) which counts the number of ways of picking out the tree vertices and pair and triad of leaves.

Without actually summing the terms for say, trees on 9 vertices, note that it can be done quite easily, as many of the 30 cycle structures cannot be symmetries of trees at all, and each of the others allow computation of their contributions as done in the last paragraph.

By the way, this sort of counting, which has applications to counting isomers in chemistry, is called Polya Theory in honor of one of its discoverers.

When counting tree patterns, bearing in mind that roughly n/e vertices will in general be leaves for large n, it will not be unusual for two vertices to both be leaves attached to the same vertex, and this means there is a symmetry between them. Thus the contributions from symmetries make a contribution here.

Often large structures have very little symmetry and the contribution from the identity term (total number of labeled patterns divided by the order of the symmetry group) is close to the number of patterns.

Exercises

Exercise1 Prove that the sum of a function f over all patterns is equivalent to the average over all group elements g of the sum of f over the patterns stabilized by g, given that f is constant on the orbit.

Exercise 2 In the necklace example given in section 18.2, how many colorings have exactly 2 red beads and 4 green beads with 7 colors all together?

Exercise 3 Write out the general formula for necklaces of length 23 analogous to the one given for necklaces of length 10 with powers of sums of powers of t_s, and compute how many patterns of colorings there are all together for 5 colors of beads.

Exercise 4 How many tree patterns are there with 7 vertices? Compute this directly and also by looking at all cycle structures in the symmetric group on 7 vertices, and counting the average number of trees stabilized by the group elements.

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