

**Problem Set 5 due at lecture on Wednesday, May 10, 2006**

**I. Problems from H1 on Fourier transform**, Ch. 10.3: 5, 6, 7, 11; and **linear diffusion**, Ch. 10.4: 3, 4.

**II. Additional problems.**

1. **Time-dependent Diffusion.** In many situations, the diffusivity is time-dependent,  $D(t) = D_o f(t) \geq 0$ , which leads to a linear PDE with a non-constant coefficient,

$$\frac{\partial c}{\partial t} = D_o f(t) \frac{\partial^2 c}{\partial x^2}. \quad (1)$$

- (a) Show that the PDE can be transformed into the diffusion equation (with a constant diffusivity) by changing to a modified “time” variable,

$$\tau = \int_0^t f(t') dt', \quad (2)$$

and find the Green function, which solves Eq. (1) with  $c(x, 0) = \delta(x)$ .

- (b) Solve for the concentration in a half-space ( $x > 0$ ) of initially uniform concentration, when the diffusion constant decays exponentially with time, substrate)

$$\frac{\partial c}{\partial t} = D_o e^{-rt} \frac{\partial^2 c}{\partial x^2}, \quad c(0, t) = 0, \quad c(x, 0) = c_o. \quad (3)$$

Problems like this arise in geology when determining the age of a rock formed from cooling magma.

2. **Regularized Traffic Flow.** Consider the “looking-ahead” theory of traffic flow:

$$\frac{\partial \rho}{\partial t} + c(\rho) \frac{\partial \rho}{\partial x} = \nu \frac{\partial^2 \rho}{\partial x^2} \quad (4)$$

$$c(\rho) = u_{max} \left( 1 - \frac{2\rho}{\rho_j} \right) \quad (5)$$

where  $\nu > 0$  (analogous to the viscosity in fluid mechanics) prevents shock formation and thus “regularizes” the standard model. Using the Cole-Hopf transformation and the Green function for the diffusion equation, derive an integral expression for the general solution,  $\rho(x, t)$ , for an initial traffic density,  $\rho_o(x) = \rho(x, 0)$ .

3. **Nonlinear Diffusion in a Porous Medium.** Consider the (dimensionless) “porous medium equation”,

$$\frac{\partial \rho}{\partial t} = \frac{\partial^2 \rho^{1+\alpha}}{\partial x^2}. \quad (6)$$

where  $\alpha \geq 0$ , which describes, e.g., the height of a groundwater flow.

- (a) Assume  $\rho(\pm\infty) = 0$  and prove global conservation,  $\frac{\partial}{\partial t} \int_{-\infty}^{\infty} \rho dx = 0$ .
- (b) Look for a similarity solution of the form,

$$\rho(x, t) = \frac{1}{t^\beta} F\left(\frac{x}{t^\beta}\right). \quad (7)$$

What is the scaling exponent  $\beta$ , and how does it compare with the case of linear diffusion ( $\alpha = 0$ )? Derive an ODE for the scaling function,  $F(\xi)$ .

- (c) Show that a continuous, positive solution for  $F(\xi)$  must have singular “fronts” at  $\xi = \pm\xi_o$ , where  $F'(\xi)$  is discontinuous and beyond which  $F(\xi) = 0$ , for  $|\xi| > \xi_o$ . Determine  $\xi_o$  such that  $\int_{-\infty}^{\infty} \rho dx = 1$  for all times. Sketch the solution  $\rho(x, t)$  at several times, and compare with the case of linear diffusion ( $\alpha = 0$ ).