18.311 Principles of Applied Mathematics, Spring 2006, Prof. Bazant

Problem Set 3 due at lecture on Monday, March 20, 2006

I. Problems on **traffic flow** from H2 – 77.1, 77.2, 78.2, 80.1, 82.13 and on **1st order quasilinear PDE** from H1 – 12.6.8, 12.6.10.

II. Additional problems.

1. **Traffic-Flow Signaling.** Suppose that we would like to control the flux of cars entering a tunnel by installing a traffic signal at the entrance $x = 0$, which would attempt to control the entering density of cars, $\rho(0, t)$ for all times $t \geq 0$ according to $\rho(x,\, 0) = 0$ and

$$\rho(0,\, t) = \begin{cases} \dfrac{\rho_*\, t}{\tau} & \text{for } t \leq \tau\,, \\ \rho_* & \text{for } t \geq \tau\,, \end{cases}$$

where $\rho_* = \dfrac{1}{2}\,\rho_j$, and ρ_j is the jam concentration. Solve for the traffic flow for $x > 0$, $t > 0$, assuming a linear velocity-density relation, $u = u_{max}(1 - \rho/\rho_j)$. If $\rho_* > \dfrac{1}{2}\,\rho_j = \rho_m$, is it still possible to prescribe (i.e.: control) the concentration $\rho(0,\, t)$ with any sort of signaling protocol? If not, why?

2. **Shock Dynamics**. Assume $u = u_m(1 - (\rho/\rho_j)^2)$ and revisit the problem discussed in class and H2 of a temporary flow of traffic initially stopped behind a red light at $x = 0$,

$$\rho(x, 0) = \begin{cases} \rho_j & x < 0 \\ 0 & x > 0 \end{cases}$$

and then allowed to flow until the light turns red again, $\rho(0^-, t) = \rho_j$ and $\rho(0^+, t) = 0$ for $t > T$, creating two shocks, which move in the expansion fan. Focus on the shock behind the light, which causes cars to suddenly stop.

   (a) Derive an ODE for the shock locus $x_s(t)$ in space-time.

   (b) Explain why the shock strength tends to zero, $[\rho] \to 0$, and use this fact to approximate the shock locus at long times, $t \gg T$, by using the "weak shock" approximation from class (and H2 77.3).