18.311 Principles of Applied Mathematics, Spring 2006, Prof. Bazant

Problem Set 4 due at lecture on Monday, April 24, 2005

I. Problems from H1 on **dispersive waves**. Ch. 14.2: 1, 2, 9; 14.6: 3, 4.

II. Additional problems on **shallow-water waves**.

1. **Shallow-water waves on an incline plane.** Consider a thin water layer of height $h(x,t)$ and density ρ_0 flowing down an inclined plane of angle θ (from horizontal), where x is the distance down the plane. As with "inertial" river waves, neglect surface tension and assume $1d$ compressible, inviscid flow with a depth-averaged fluid velocity, $u(x,t)$, and an effective density, $\rho(x,t) = \rho_0 h(x,t)/h_0$. The pressure is the depth-averaged gravitational energy (per unit volume): $p = \rho_0(\tilde{g}h - s\tilde{g}x)$, where $\tilde{g} = g\cos\theta$, $s = \tan\theta$, and $g = 9.8$ m^2/s is the gravitational acceleration. Finally, make the *Boussinesq approximation* of nearly constant density, $\rho = \rho_0$, in the momentum-conservation equation, while still allowing for a variable pressure, $p(\rho, x)$, and derive the (inviscid) shallow-water PDEs:

$$\begin{aligned} h_t + uh_x + hu_x &= 0 \\ u_t + uu_x + \tilde{g}h_x &= \tilde{g}s \end{aligned}$$

2. **Nonlinear shallow-water waves on a flat surface.** In the shallow-water PDEs above with $\theta = 0$, consider characteristic curves, $\xi(x,t)$ =constant, along which $h(x,t) = F(\xi(x,t))$ and $u(x,t) = G(\xi(x,t))$. Show that the two Riemann invariants,

$$\Gamma_\pm = u \pm 2\sqrt{gh}$$

are constant along characteristics, which move with corresponding velocities,

$$\frac{dx}{dt} = u \pm \sqrt{gh}.$$

3. **Simple shallow-water waves due to a receding wall.** Consider a half plane of flat, inviscid shallow water, $x > 0$, which is initially at rest, $h(x,0) = h_0$ and $u(x,0) = 0$. Suppose that the left wall at $x = 0$ moves to the left with velocity, $U_w < 0$, for $t > 0$.

   (a) Find the *simple-wave solution* with only right-moving waves ($\Gamma_- =$ constant everywhere) for $h(x,t)$ and $u(x,t)$.

   (b) What is the maximum wall speed $|U_w|$ where the simple-wave solution breaks down? Describe roughly what happens above this speed.