Lecture 1
Introduction, Basic Linear Algebra

MIT 18.335J / 6.337J

Introduction to Numerical Methods

Per-Olof Persson

September 7, 2006

Matrix-Vector Multiplication

- Matrix-vector product $b = Ax$

$$b_i = \sum_{j=1}^{n} a_{ij} x_j, \qquad i = 1, \ldots, m$$

- *Linear* mapping $x \mapsto Ax$

$$A(x + y) = Ax + Ay$$
$$A(\alpha x) = \alpha Ax$$

- Every linear map can be expressed as a matrix-vector product

Linear Combination of Columns

- Columns $a_1, a_2, \ldots, a_n$ of A:

$$A = \left[a_1 \middle| a_2 \middle| \cdots \middle| a_n \right]$$

- Alternative view of matrix-vector product:

$$b = Ax = \sum_{j=1}^{n} x_j a_j = x_1 \begin{bmatrix} a_1 \end{bmatrix} + x_2 \begin{bmatrix} a_2 \end{bmatrix} + \cdots + x_n \begin{bmatrix} a_n \end{bmatrix}$$

- b is a *linear combination of the columns of* A

Matrix-Matrix Multiplication

- Matrix-matrix product $B = AC$

$$b_{ij} = \sum_{k=1}^{m} a_{ik} c_{kj}$$

- Matrix-vector product for each column of C
- Each column of B is a linear combination of the columns of A

Range, Nullspace, and Rank

- The *range* or *column space* of A:

$$\begin{aligned}\mathrm{range}(A) &= \text{All linear combinations of the columns of } A \\ &= \text{The space spanned by the columns of } A \\ &= \text{All vectors that can be expressed as } Ax\end{aligned}$$

- The *nullspace* of A:

$$\mathrm{null}(A) = \text{All solutions to } Ax = 0$$

- Dimension of space = number of vectors in a basis
- The *column rank* of A is the dimension of the column space $\mathrm{range}(A)$
- column rank = row rank = rank

Matrix Inverse

- *Nonsingular* or *invertible* matrix = square matrix with full rank
- The *inverse* A^{-1} of A satisfies

$$AA^{-1} = A^{-1}A = I$$

- Change of basis:

$$\begin{aligned} x = A^{-1}b &= \text{solution to } Ax = b \\ &= \text{the vector of coefficients of the expansion of } b \\ &\quad \text{in the basis of columns of } A \end{aligned}$$