MIT 18.335, Fall 2006: Homework 4

Due November 2

1. Trefethen/Bau 20.1

2. Trefethen/Bau 20.3

3. Trefethen/Bau 23.2

4. Consider a symmetric positive definite m-by-m matrix A with lower and upper bandwidth p (see homework 2 for definitions). Now we are concerned about the memory usage and store the matrices in a *banded* storage format. Instead of storing A, we use an m-by-$(p+1)$ matrix A^b with

$$A^b_{i,j-i+1} = A_{ij} \text{ for } j \geq i. \tag{1}$$

That is, we store only the main diagonal and the first p superdiagonals.

(a) Rewrite the Cholesky factorization algorithm in terms of A^b instead of A (using pseudo-code as in Algorithm 23.1 in the textbook), to produce a Cholesky factor R^b in banded format. Determine the operation count for the modified algorithm.

Hint: You might find it easier to first rewrite the algorithm to take sparsity into account, but still in terms of A. Then replace A with A^b and change the indices according to (1).

(b) Write pseudo-code for these two algorithms:

(1) Solve $Rx = f$ by back substitution

(2) Solve $R^T x = f$ by forward substitution

where R is stored in a banded form R^b. Determine the operation counts.

(c) Implement the three algorithms in MATLAB, using the following function headers (and the corresponding function names):

```
function Rb=bandchol(Ab)
function x=bandbacksub(Rb,f)
function x=bandforwardsub(Rb,f)
```

Verify your code by running `bandtest` (on the web and on the next page). This script solves a PDE using your functions and compares with MATLAB. Make sure that the error is small.

You might also want to experiment with different p values, e.g. to verify your operation count in **(a)**, or just for fun (do not hand in).

5. Trefethen/Bau 24.1

6. (Z. Bai) Suppose that A is *normal*; i.e., $AA^* = A^*A$. Show that if A is also triangular, it must be diagonal. Use this to show that an n-by-n matrix is normal if and only if it has n orthonormal eigenvectors. Hint: Show that A is normal if and only if its Schur form is normal.

Turn the page $\Longrightarrow$

7. Let λ be a simple eigenvalue of A with right eigenvector x and left eigenvector y (that is, $Ax = \lambda x$ and $y^*A = y^*\lambda$), normalized so that $\|x\|_2 = \|y\|_2 = 1$. Define the *eigenvalue condition number* by $\kappa(\lambda, A) = 1/|y^*x|$.

(a) Let $\lambda + \delta\lambda$ be the corresponding eigenvalue of $A + \delta A$. Show that

$$\delta\lambda = \frac{y^*\delta A x}{y^*x} + \mathcal{O}(\|\delta A\|_2^2), \quad \text{or} \quad |\delta\lambda| \le \kappa(\lambda, A)\|\delta A\|_2 + \mathcal{O}(\|\delta A\|_2^2).$$

(b) Consider an m-by-m Jordan block

$$J = \begin{bmatrix} \lambda & 1 & & \\ & \lambda & \ddots & \\ & & \ddots & 1 \\ & & & \lambda \end{bmatrix}.$$

Calculate the eigenvalue condition number of J. Demonstrate the sensitivity of λ with a simple MATLAB example (for $m = 16$).

(c) What are the eigenvalue condition numbers of a symmetric (or normal) matrix?

8. Trefethen/Bau 25.3

MATLAB Codes (also on the webpage)

bandtest.m:

```
% Initialize banded matrix and RHS for Poisson equation on unit square
p=20; m=p^2;
Ab=zeros(m,p+1); Ab(:,1)=4; Ab(:,2)=-1; Ab(p:p:m,2)=0; Ab(:,p+1)=-1;
f=ones(m,1)/(p+1)^2;

% Solve using banded solver
Rb=bandchol(Ab);
y=bandforwardsub(Rb,f);
x=bandbacksub(Rb,y);

% Solve using MATLAB
A=spdiags(Ab(:,[1,2,p+1]),-[0,1,p],m,m); A=A+tril(A,-1)';
x0=A\f;

% Plot and compare solutions
surf(reshape(x,p,p)); view(2); axis equal;
shading interp; colorbar; set(gcf,'rend','z');
error=norm(x-x0)
```