Lecture 2
Orthogonal Vectors and Matrices, Norms

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Introduction to Numerical Methods

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Transpose and Adjoint

- For real A, the *transpose* of A is obtained by interchanging rows/columns

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} \Longrightarrow A^T = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \end{bmatrix}$$

- The *adjoint* or *hermitian conjugate* also takes complex conjugate

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} \Longrightarrow A^* = \begin{bmatrix} \overline{a}_{11} & \overline{a}_{21} & \overline{a}_{31} \\ \overline{a}_{12} & \overline{a}_{22} & \overline{a}_{32} \end{bmatrix}$$

- If real $A = A^T$, then A is *symmetric*. If $A = A^*$, then A is *hermitian*.

Inner Product

- *Inner product* of two column vectors $x, y \in \mathbb{C}^m$

$$x^* y = \sum_{i=1}^{m} \overline{x_i} y_i$$

- *Euclidean length* of x

$$\|x\| = \sqrt{x^* x} = \left(\sum_{i=1}^{m} |x_i|^2 \right)^{1/2}$$

- Angle α between x, y

$$\cos \alpha = \frac{x^* y}{\|x\| \|y\|}$$

In MATLAB

Quantity	**MATLAB Syntax**	**Comment**
Transpose A^T	`A.'`	Transpose only
Adjoint A^*	`A'`	Transpose + complex conjugate
Inner product x^*y	`x'*y` or `dot(x,y)`	`'*` assumes column vectors
Length $\|x\|$	`sqrt(x'*x)` or `norm(x)`	`'*` assumes column vector

Orthogonal Vectors

- The vectors $x, y \in \mathbb{R}^m$ are *orthogonal* if

$$x^* y = 0$$

- The sets of vectors X, Y are orthogonal if

$$\text{every } x \in X \text{ is orthogonal to every } y \in Y$$

- A set of (nonzero) vectors S is orthogonal if

$$\text{vectors pairwise orthogonal, i.e., for } x, y \in S, x \neq y \Rightarrow x^* y = 0$$

and *orthonormal* if, in addition,

$$\text{every } x \in S \text{ has } \|x\| = 1$$

Orthogonal and Unitary Matrices

- A square matrix $Q \in \mathbb{C}^{m \times m}$ is *unitary* (*orthogonal* in real case), if

$$Q^* = Q^{-1}$$

- For unitary Q

$$Q^* Q = I, \text{ or } q_i^* q_j = \delta_{ij}$$

- Interpretation of unitary-times-vector product:

$$\begin{aligned} x = Q^* b &= \text{solution to } Qx = b \\ &= \text{the vector of coefficients of the expansion of } b \\ &\quad\ \text{in the basis of columns of } Q \end{aligned}$$

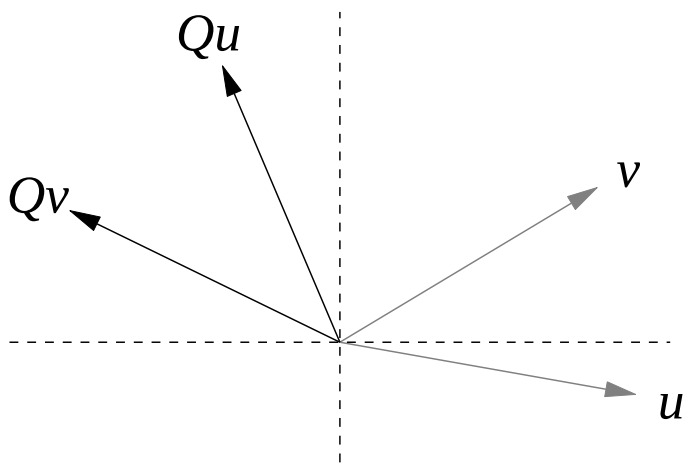
Preservation of Geometry Structure

- Inner product is preserved under multiplication by unitary Q

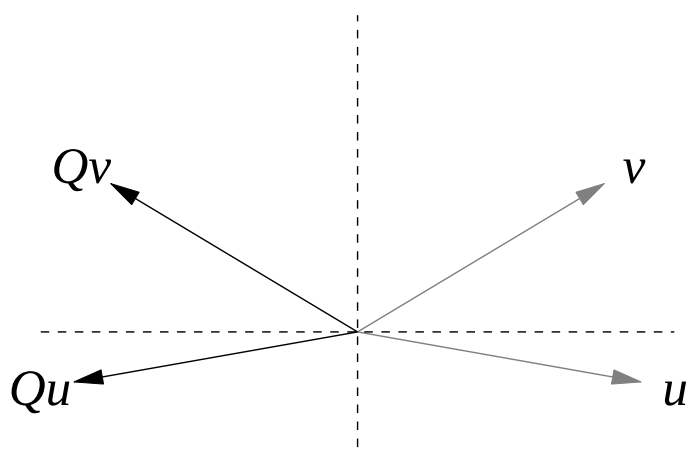
$$(Qx)^*(Qy) = x^*Q^*Qy = x^*y$$

- Therefore, *lengths of vectors* and *angles between vectors* are preserved
- A real orthogonal Q is either a rigid rotation or reflection

Rotation



Reflection



Vector Norms

- A *norm* is a function $\|\cdot\| : \mathbb{C}^m \to \mathbb{R}$ satisfying

$$
\begin{aligned}
&(1) \quad \|x\| \geq 0, \text{ and } \|x\| = 0 \text{ only if } x = 0 \\
&(2) \quad \|x + y\| \leq \|x\| + \|y\| \\
&(3) \quad \|\alpha x\| = |\alpha| \, \|x\|
\end{aligned}
$$

- Example: The Euclidean length function

$$\|x\|_2 = \sqrt{x^* x}$$

- $\|\cdot\|_2$ is a special case of the p-norms

$$\|x\|_p = \left(\sum_{i=1}^{m} |x_i|^p \right)^{1/p} \qquad (1 \leq p < \infty)$$

Examples of Vector Norms

$$\|x\|_1 = \sum_{i=1}^{m} |x_i|$$

$$\|x\|_2 = \left(\sum_{i=1}^{m} |x_i|^2 \right)^{1/2} = \sqrt{x^* x}$$

$$\|x\|_\infty = \max_{1 \le i \le m} |x_i|$$

$$\|x\|_W = \|Wx\|_2 = \left(\sum_{i=1}^{m} |w_i x_i|^2 \right)^{1/2}$$

Verification of Norm Conditions

- Show that $\|x\|_1 = \sum_{i=1}^{m} |x_i|$ is a norm

(1)

$$\|x\|_1 = |x_1| + |x_2| + \cdots + |x_m| \geq 0, \text{ equality only if } x = 0$$

(2)

$$\begin{aligned}\|x+y\|_1 &= \sum_{i=1}^{m} |x_i + y_i| \leq \sum_{i=1}^{m} \big(|x_i| + |y_i| \big) \\ &= \sum_{i=1}^{m} |x_i| + \sum_{i=1}^{m} |y_i| = \|x\|_1 + \|y\|_1\end{aligned}$$

(3)

$$\|\alpha x\|_1 = \sum_{i=1}^{m} |\alpha x_i| = \sum_{i=1}^{m} |\alpha|\,|x_i| = |\alpha| \sum_{i=1}^{m} |x_i| = |\alpha|\,\|x\|_1$$

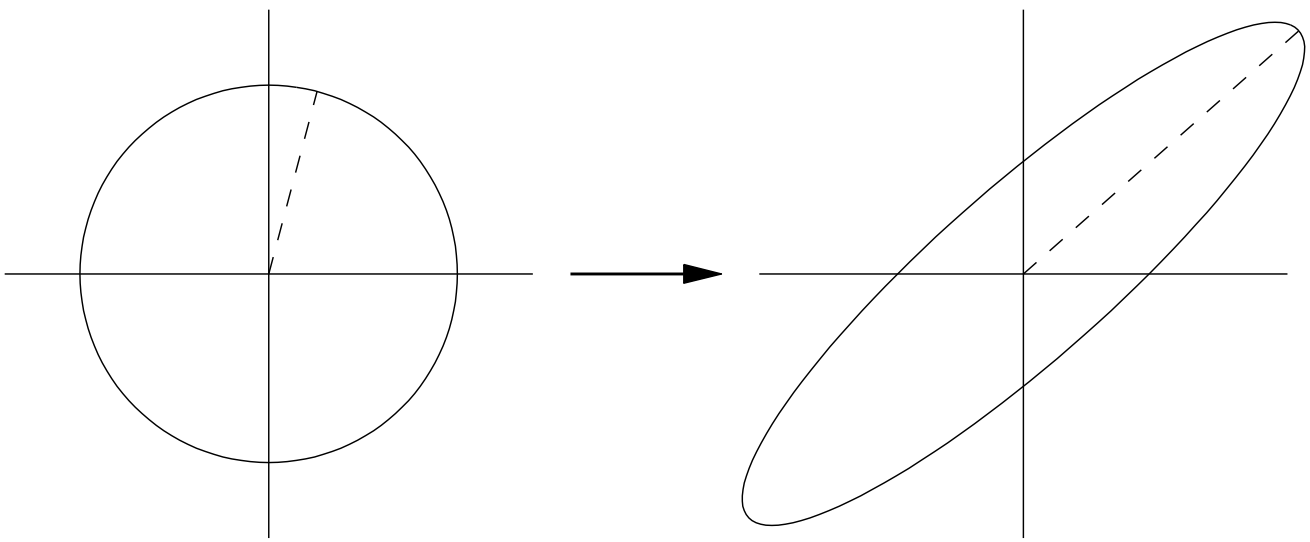
Induced Matrix Norms

- For a matrix $A \in \mathbb{C}^{m\times n}$, the *induced matrix norm* is

$$\|A\|_{(m,n)} = \sup_{\substack{x\in\mathbb{C}^n \\ x\neq 0}} \frac{\|Ax\|_{(m)}}{\|x\|_{(n)}} = \sup_{\substack{x\in\mathbb{C}^n \\ \|x\|_{(n)}=1}} \|Ax\|_{(m)}$$

  where $\|\cdot\|_{(n)}$ and $\|\cdot\|_{(m)}$ are given vector norms

- The "maximum stretching" by A



Examples of Matrix Norms

$$\|A\|_1 = \max_{1\le j\le n} \sum_{i=1}^{m} |a_{ij}|$$ "maximum column sum"

$$\|A\|_\infty = \max_{1\le i\le m} \sum_{j=1}^{n} |a_{ij}|$$ "maximum row sum"

$$\|A\|_F = \left(\sum_{i=1}^{m} \sum_{j=1}^{n} |a_{ij}|^2 \right)^{1/2}$$ The Frobenius norm

$$\|A\|_2 = \sqrt{\lambda_{\max}(A^* A)}$$ More later

Properties of Matrix Norms

- *Bound on Matrix Product* - Induced norms and Frobenius norm satisfy

$$\|AB\| \leq \|A\|\|B\|$$

  but some matrix norms do not!

- *Invariance under Unitary Multiplication* - For $A \in \mathbb{C}^{m\times n}$ and unitary $Q \in \mathbb{C}^{m\times m}$, we have

$$\|QA\|_2 = \|A\|_2, \qquad \|QA\|_F = \|A\|_F$$

  Proof. Since $\|Qx\|_2 = \|x\|_2$ (inner product is preserved), the first result follows from the definition of induced norm. For the Frobenius norm,

$$\begin{aligned}\|QA\|_F &= \sqrt{\mathrm{tr}((QA)^*(QA))} = \sqrt{\mathrm{tr}(A^*Q^*QA)} \\ &= \sqrt{\mathrm{tr}(A^*A)} = \|A\|_F\end{aligned}$$

In MATLAB

Quantity	MATLAB Syntax
$\|x\|_1$	`sum(abs(x))` or `norm(x,1)`
$\|x\|_2$	`sqrt(x'*x)` or `norm(x)`
$\|x\|_p$	`sum(abs(x).^p).^(1/p)` or `norm(x,p)`
$\|x\|_\infty$	`max(abs(x))` or `norm(x,inf)`
$\|A\|_1$	`max(sum(abs(A),1))` or `norm(A,1)`
$\|A\|_2$	`norm(A)`
$\|A\|_\infty$	`max(sum(abs(A),2))` or `norm(A,inf)`
$\|A\|_F$	`sqrt(A(:)'*A(:))` or `norm(A,'fro')`