# Lecture 2
# Orthogonal Vectors and Matrices, Norms

MIT 18.335J / 6.337J

Introduction to Numerical Methods

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September 12, 2006



## Transpose and Adjoint

- For real $A$, the *transpose* of $A$ is obtained by interchanging rows/columns

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} \Longrightarrow A^T = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \end{bmatrix}$$

- The *adjoint* or *hermitian conjugate* also takes complex conjugate

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} \Longrightarrow A^* = \begin{bmatrix} \overline{a}_{11} & \overline{a}_{21} & \overline{a}_{31} \\ \overline{a}_{12} & \overline{a}_{22} & \overline{a}_{32} \end{bmatrix}$$

- If real $A = A^T$, then $A$ is *symmetric*. If $A = A^*$, then $A$ is *hermitian*.



## Inner Product

- *Inner product* of two column vectors $x, y \in \mathbb{C}^m$

$$x^* y = \sum_{i=1}^{m} \overline{x_i} y_i$$

- *Euclidean length* of $x$

$$\|x\| = \sqrt{x^* x} = \left( \sum_{i=1}^{m} |x_i|^2 \right)^{1/2}$$

- Angle $\alpha$ between $x, y$

$$\cos \alpha = \frac{x^* y}{\|x\| \|y\|}$$



## In MATLAB

| Quantity | MATLAB Syntax | Comment |
|---|---|---|
| Transpose $A^T$ | `A.'` | Transpose only |
| Adjoint $A^*$ | `A'` | Transpose + complex conjugate |
| Inner product $x^*y$ | `x'*y` or `dot(x,y)` | `'*` assumes column vectors |
| Length $\|x\|$ | `sqrt(x'*x)` or `norm(x)` | `'*` assumes column vector |



## Orthogonal Vectors

- The vectors $x, y \in \mathbb{R}^m$ are *orthogonal* if

$$x^* y = 0$$

- The sets of vectors $X, Y$ are orthogonal if

every $x \in X$ is orthogonal to every $y \in Y$

- A set of (nonzero) vectors $S$ is orthogonal if

vectors pairwise orthogonal, i.e., for $x, y \in S, x \neq y \Rightarrow x^* y = 0$

and *orthonormal* if, in addition,

every $x \in S$ has $\|x\| = 1$



## Orthogonal and Unitary Matrices

- A square matrix $Q \in \mathbb{C}^{m \times m}$ is *unitary* (*orthogonal* in real case), if

$$Q^* = Q^{-1}$$

- For unitary $Q$

$$Q^* Q = I, \text{ or } q_i^* q_j = \delta_{ij}$$

- Interpretation of unitary-times-vector product:

$x = Q^* b =$ solution to $Qx = b$

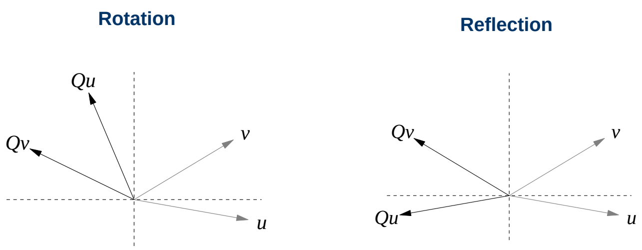
$=$ the vector of coefficients of the expansion of $b$ in the basis of columns of $Q$

## Preservation of Geometry Structure

- Inner product is preserved under multiplication by unitary $Q$

$$(Qx)^*(Qy) = x^*Q^*Qy = x^*y$$

- Therefore, *lengths of vectors* and *angles between vectors* are preserved
- A real orthogonal $Q$ is either a rigid rotation or reflection






## Vector Norms

- A *norm* is a function $\|\cdot\| : \mathbb{C}^m \to \mathbb{R}$ satisfying

$$\begin{aligned}
&(1) \quad \|x\| \ge 0, \text{ and } \|x\| = 0 \text{ only if } x = 0 \\
&(2) \quad \|x+y\| \le \|x\| + \|y\| \\
&(3) \quad \|\alpha x\| = |\alpha|\,\|x\|
\end{aligned}$$

- Example: The Euclidean length function

$$\|x\|_2 = \sqrt{x^*x}$$

- $\|\cdot\|_2$ is a special case of the $p$-norms

$$\|x\|_p = \left(\sum_{i=1}^m |x_i|^p\right)^{1/p} \qquad (1 \le p < \infty)$$



## Examples of Vector Norms

$$\begin{aligned}
\|x\|_1 &= \sum_{i=1}^m |x_i| \\
\|x\|_2 &= \left(\sum_{i=1}^m |x_i|^2\right)^{1/2} = \sqrt{x^*x} \\
\|x\|_\infty &= \max_{1\le i\le m} |x_i| \\
\|x\|_W &= \|Wx\|_2 = \left(\sum_{i=1}^m |w_i x_i|^2\right)^{1/2}
\end{aligned}$$



## Verification of Norm Conditions

- Show that $\|x\|_1 = \sum_{i=1}^m |x_i|$ is a norm

(1)

$$\|x\|_1 = |x_1| + |x_2| + \cdots + |x_m| \ge 0, \text{ equality only if } x = 0$$

(2)

$$\begin{aligned}
\|x+y\|_1 &= \sum_{i=1}^m |x_i + y_i| \le \sum_{i=1}^m \big(|x_i| + |y_i|\big) \\
&= \sum_{i=1}^m |x_i| + \sum_{i=1}^m |y_i| = \|x\|_1 + \|y\|_1
\end{aligned}$$

(3)

$$\|\alpha x\|_1 = \sum_{i=1}^m |\alpha x_i| = \sum_{i=1}^m |\alpha|\,|x_i| = |\alpha| \sum_{i=1}^m |x_i| = |\alpha|\,\|x\|_1$$



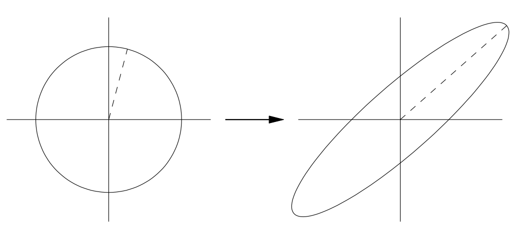
## Induced Matrix Norms

- For a matrix $A \in \mathbb{C}^{m\times n}$, the *induced matrix norm* is

$$\|A\|_{(m,n)} = \sup_{\substack{x\in\mathbb{C}^n \\ x\ne 0}} \frac{\|Ax\|_{(m)}}{\|x\|_{(n)}} = \sup_{\substack{x\in\mathbb{C}^n \\ \|x\|_{(n)}=1}} \|Ax\|_{(m)}$$

where $\|\cdot\|_{(n)}$ and $\|\cdot\|_{(m)}$ are given vector norms

- The "maximum stretching" by $A$





## Examples of Matrix Norms

| | |
|---|---|
| $\|A\|_1 = \max_{1\le j\le n} \sum_{i=1}^m \lvert a_{ij}\rvert$ | "maximum column sum" |
| $\|A\|_\infty = \max_{1\le i\le m} \sum_{j=1}^n \lvert a_{ij}\rvert$ | "maximum row sum" |
| $\|A\|_F = \left(\sum_{i=1}^m \sum_{j=1}^n \lvert a_{ij}\rvert^2\right)^{1/2}$ | The Frobenius norm |
| $\|A\|_2 = \sqrt{\lambda_{\max}(A^*A)}$ | More later |

## Properties of Matrix Norms

- *Bound on Matrix Product* - Induced norms and Frobenius norm satisfy

$$\|AB\| \leq \|A\|\|B\|$$

but some matrix norms do not!

- *Invariance under Unitary Multiplication* - For $A \in \mathbb{C}^{m \times n}$ and unitary $Q \in \mathbb{C}^{m \times m}$, we have

$$\|QA\|_2 = \|A\|_2, \qquad \|QA\|_F = \|A\|_F$$

*Proof.* Since $\|Qx\|_2 = \|x\|_2$ (inner product is preserved), the first result follows from the definition of induced norm. For the Frobenius norm,

$$\|QA\|_F = \sqrt{\mathrm{tr}((QA)^*(QA))} = \sqrt{\mathrm{tr}(A^*Q^*QA)} = \sqrt{\mathrm{tr}(A^*A)} = \|A\|_F$$



## In MATLAB

| **Quantity** | **MATLAB Syntax** |
|---|---|
| $\|x\|_1$ | `sum(abs(x))` or `norm(x,1)` |
| $\|x\|_2$ | `sqrt(x'*x)` or `norm(x)` |
| $\|x\|_p$ | `sum(abs(x).^p).^(1/p)` or `norm(x,p)` |
| $\|x\|_\infty$ | `max(abs(x))` or `norm(x,inf)` |
| $\|A\|_1$ | `max(sum(abs(A),1))` or `norm(A,1)` |
| $\|A\|_2$ | `norm(A)` |
| $\|A\|_\infty$ | `max(sum(abs(A),2))` or `norm(A,inf)` |
| $\|A\|_F$ | `sqrt(A(:)'*A(:))` or `norm(A,'fro')` |