

# **Lecture 17**

## **Other Eigenvalue Algorithms**

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Introduction to Numerical Methods

Per-Olof Persson

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# The Jacobi Algorithm

- Diagonalize  $2 \times 2$  real symmetric matrix by a *Jacobi rotation*:

$$J^T \begin{bmatrix} a & d \\ d & b \end{bmatrix} J = \begin{bmatrix} \neq 0 & 0 \\ 0 & \neq 0 \end{bmatrix}$$

where

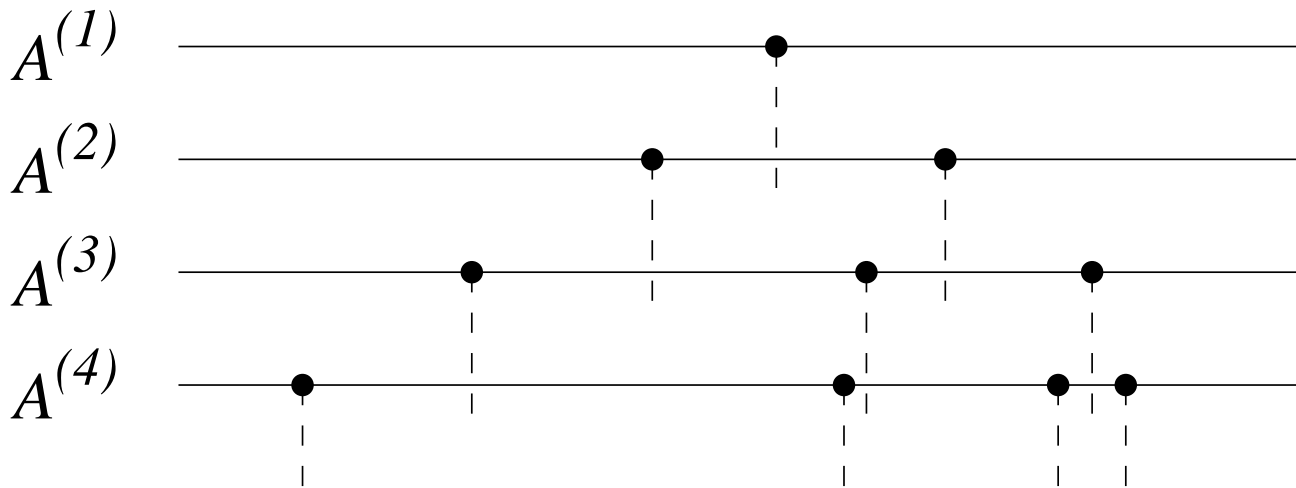
$$J = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix}, \quad \tan(2\theta) = \frac{2d}{b-a}$$

- Iteratively apply transformation to 2 rows and 2 columns of  $A \in \mathbb{R}^{m \times m}$
- Loop over all pairs of rows/columns, quadratic convergence
- $O(m^2)$  steps,  $O(m)$  operations per step  $\implies O(m^3)$  operation count

# The Method of Bisection

- Idea: Search the real line for roots of  $p(x) = \det(A - xI)$
- Finding roots from coefficients highly unstable, but  $p(x)$  could be computed by elimination
- Important property: Eigenvalues of principal upper-left square submatrices  $A^{(1)}, \dots, A^{(m)}$  *interlace*

$$\lambda_j^{(k+1)} < \lambda_j^{(k)} < \lambda_{j+1}^{(k+1)}$$



# The Method of Bisection

- Because of the interlacing property: The number of negative eigenvalues of  $A$  equals the number of sign changes in the *Sturm sequence*

$$1, \det(A^{(1)}), \det(A^{(2)}), \dots, \det(A^{(m)})$$

- Shift  $A$  to get number of eigenvalues in  $(-\infty, b)$  and twice for  $[a, b)$
- Three-term recurrence for the determinants:

$$\det(A^{(k)}) = a_k \det(A^{(k-1)}) - b_{k-1}^2 \det(A^{(k-2)})$$

- With shift  $xI$  and  $p^{(k)}(x) = \det(A^{(k)} - xI)$ :

$$p^{(k)}(x) = (a_k - x)p^{(k-1)}(x) - b_{k-1}^2 p^{(k-2)}(x)$$

- $O(m \log(\epsilon_{\text{machine}}))$  flops per eigenvalue, always high relative accuracy

# The Divide-and-Conquer Algorithm

- Split symmetric tridiagonal  $T$  into submatrices:

$$T = \begin{array}{|c|c|} \hline T_1 & \\ \hline \beta & T_2 \\ \hline \end{array} = \begin{array}{|c|c|} \hline \hat{T}_1 & \\ \hline & \hat{T}_2 \\ \hline \end{array} + \begin{array}{|c|c|} \hline & \\ \hline \beta & \beta \\ \hline \beta & \beta \\ \hline \end{array}$$

- The sum of a  $2 \times 2$  block-diagonal matrix and a rank-one correction
- Split  $T$  in equal sizes and compute eigenvalues of  $\hat{T}_1, \hat{T}_2$  recursively
- Solve nonlinear problem to get eigenvalues of  $T$  from those of  $\hat{T}_1, \hat{T}_2$

# The Divide-and-Conquer Algorithm

- Suppose diagonalizations  $\hat{T}_1 = Q_1 D_1 Q_1^T$  and  $\hat{T}_2 = Q_2 D_2 Q_2^T$  have been computed. We then have

$$T = \begin{bmatrix} Q_1 & \\ & Q_2 \end{bmatrix} \left( \begin{bmatrix} D_1 & \\ & D_2 \end{bmatrix} + \beta z z^T \right) \begin{bmatrix} Q_1^T & \\ & Q_2^T \end{bmatrix}$$

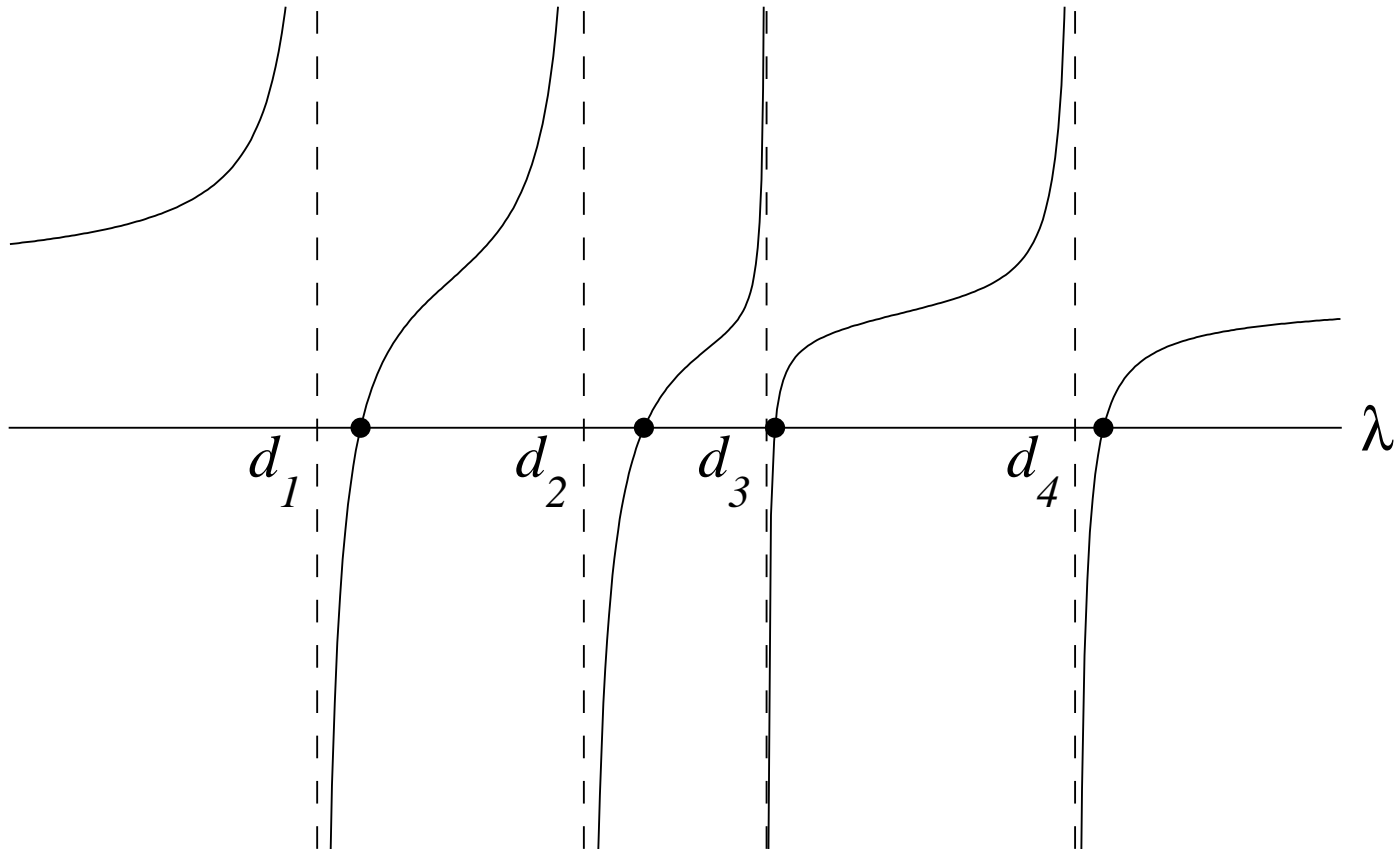
with  $z^T = (q_1^T, q_2^T)$ , where  $q_1^T$  is last row of  $Q_1$  and  $q_2^T$  is first row of  $Q_2$

- This is a similarity transformation  $\implies$  Find eigenvalues of diagonal matrix plus rank-one correction

# The Divide-and-Conquer Algorithm

- Eigenvalues of  $D + ww^T$  are the roots of the rational function

$$f(\lambda) = 1 + \sum_{j=1}^m \frac{w_j^2}{d_j - \lambda}$$



# The Divide-and-Conquer Algorithm

- Solve the *secular equation*  $f(\lambda) = 0$  with nonlinear solver
- $O(m)$  flops per root,  $O(m^2)$  flops for all roots
- Total cost for divide-and-conquer algorithm:

$$O\left(m^2 + 2\frac{m^2}{2^2} + 4\frac{m^2}{4^2} + 8\frac{m^2}{8^2} + \cdots + m\frac{m^2}{m^2}\right) = O(m^2)$$

- For computing eigenvalues only, most of the operations are spent in the tridiagonal reduction, and the constant in “Phase 2” is not important
- However, for computing eigenvectors, divide-and-conquer reduces Phase 2 to  $4m^3/3$  flops compared to  $6m^3$  for QR