Lecture 23
Arnoldi/Lanczos Iterations

MIT 18.335J / 6.337J

Introduction to Numerical Methods

Per-Olof Persson

December 5, 2006

The Arnoldi Iteration

- The Arnoldi process reduces a general, nonsymmetric A to Hessenberg form by similarity transforms: $A = QHQ^*$
- Allows for reduced factorizations by a Gram-Schmidt-style iteration instead of Householder reflections
- Let Q_n be the $m \times n$ matrix with the first n columns of Q, and consider the first n columns of $AQ = QH$, or $AQ_n = Q_{n+1}\tilde{H}_n$:

$$\begin{bmatrix} & & \\ & A & \\ & & \end{bmatrix} \begin{bmatrix} \Bigg| & & \Bigg| \\ q_1 & \cdots & q_n \\ \Bigg| & & \Bigg| \end{bmatrix} = \begin{bmatrix} \Bigg| & & \Bigg| \\ q_1 & \cdots & q_{n+1} \\ \Bigg| & & \Bigg| \end{bmatrix} \begin{bmatrix} h_{11} & \cdots & h_{1n} \\ h_{21} & & \\ & \ddots & \vdots \\ & & h_{n+1,n} \end{bmatrix}$$

The Arnoldi Algorithm

- The nth column of $AQ_n = Q_{n+1}\tilde{H}_n$ gives

$$Aq_n = h_{1n}q_1 + \cdots + h_{nn}q_n + h_{n+1,n}q_{n+1}$$

which can be used to compute q_{n+1} similarly to modified Gram-Schmidt:

Algorithm: Arnoldi Iteration

$b =$arbitrary, $q_1 = b/\|b\|$
for $n = 1, 2, 3, \dots$
  $v = Aq_n$
  for $j = 1$ **to** n
    $h_{jn} = q_j^* v$
    $v = v - h_{jn}q_j$
  $h_{n+1,n} = \|v\|$
  $q_{n+1} = v/h_{n+1,n}$

QR Factorization of Krylov Matrix

- The vectors q_j from Arnoldi are orthonormal bases of the successive *Krylov subspaces*:

$$\mathcal{K}_n = \langle b, Ab, \dots, A^{n-1}b \rangle = \langle q_1, q_2, \dots, q_n \rangle \subseteq \mathbb{C}^m$$

- Q_n is the reduced QR factorization $K_n = Q_nR_n$ of the *Krylov matrix*:

$$K_n = \begin{bmatrix} \Bigg| & \Bigg| & & \Bigg| \\ b & Ab & \cdots & A^{n-1}b \\ \Bigg| & \Bigg| & & \Bigg| \end{bmatrix}$$

- The projection of A onto this space gives $n \times n$ Hessenberg matrix $H_n = Q_n^* AQ_n$, whose eigenvalues may be good approximations of A's

Symmetric Matrices and the Lanczos Iteration

- For symmetric A, H_n reduces to tridiagonal T_n, and q_{n+1} can be computed by a three-term recurrence:

$$Aq_n = \beta_{n-1}q_{n-1} + \alpha_n q_n + \beta_n q_{n+1}$$

Algorithm: Lanczos Iteration

$\beta_0 = 0$, $q_0 = 0$, $b =$arbitrary, $q_1 = b/\|b\|$
for $n = 1, 2, 3, \dots$
  $v = Aq_n$
  $\alpha_n = q_n^T v$
  $v = v - \beta_{n-1}q_{n-1} - \alpha_n q_n$
  $\beta_n = \|v\|$
  $q_{n+1} = v/\beta_n$