3.2 The Equation $u_t + Au_x = Bu$

Take Lax–Wendroff for $u_t + Au_x = 0$ and just add the lower order term

$$v_m^{n+1} = v_m^n - \frac{\lambda}{2}A(v_{m+1}^n - v_{m-1}^n) + \frac{\lambda^2}{2}A^2(v_{m+1}^n - 2v_m^n + v_{m-1}^n) + kBv_m^n$$

The amplification factor for the scheme becomes

$$\hat{v}^{n+1}(\xi) = (G + kB)\hat{v}^n(\xi),$$

so we end up with the question when $|(G+kB)^n|$ is bounded by a constant, where $|G^n(h\xi)| \leq C$. We use Strang's idea:

$$\begin{aligned}(G+kB)^n &= G^n + k\left(G^{n-1}B + G^{n-2}BG + \ldots BG^{n-1}\right)\\ &\quad + k^2\left(G^{n-2}B^2 + G^{n-3}BGB + B^{n-4}BG^2B + \ldots + B^2G^{n-2}\right)\\ &\quad + \ldots\\ &\quad + k^n(G^0BG^0B\ldots G^0BG^0).\end{aligned}$$

Now estimate upward using $|G^n| \leq C$ to obtain

$$\begin{aligned}|(G+kB)^n| &\leq C + \binom{n}{1}kC^2|B| + \binom{n}{2}k^2C^3|B|^2 + \ldots\\ &\leq C\left(1 + \binom{n}{1}kC\cdot|B| + \binom{n}{2}k^2C^2\cdot|B|^2 + \ldots\right)\\ &\leq C(1+k|B|C)^n \leq Ce^{nk|B|C} \leq Ce^{T|B|C}\end{aligned}$$

for all $nk \leq T$.

Therefore the condition for stability is once again $|G^n| \leq C$.