

2.2 The Du Fort–Frankel Scheme

This is an example of an *explicit* and *unconditionally stable* scheme for $u_t = bu_{xx}$.

The problem with schemes like forward time, centered space is that they are stable for $b\mu = bk/h^2 \leq \frac{1}{2}$, which puts a terrible restriction $k \leq \frac{h^2}{2b}$ on the timestep. The Du Fort–Frankel scheme,

$$v_m^{n+1} - v_m^{n-1} = 2b\mu(v_{m+1}^n - (v_m^{n+1} + v_m^{n-1}) + v_{m-1}^n),$$

is a slight modification of the *unstable* Leap–Frog scheme. We rewrite the Du Fort–Frankel scheme as

$$(1 + 2b\mu)v_m^{n+1} - (1 - 2b\mu)v_m^{n-1} = 2b\mu(v_{m+1}^n + v_{m-1}^n).$$

To study the stability, we substitute $v_m^n = g^n e^{imh\xi}$ to get

$$(1 + 2b\mu)g^2 - (1 - 2b\mu) = 2b\mu(e^{ih\xi} + e^{-ih\xi})g,$$

which implies

$$g_{\pm} = \frac{2b\mu \cos(h\xi) \pm \sqrt{1 - 4b^2\mu^2 \sin^2(h\xi)}}{1 + 2b\mu}.$$

The scheme is not dissipative since $g_-(\pi) = -1$. To determine stability we consider two cases:

- $1 - 4b^2\mu^2 \geq 0 \Rightarrow |g_{\pm}| \leq \frac{2b\mu|\cos(h\xi)| + \sqrt{1}}{1 + 2b\mu} \leq \frac{2b\mu + 1}{1 + 2b\mu} = 1.$
- $1 - 4b^2\mu^2 < 0 \Rightarrow |g_{\pm}|^2 = \frac{(2b\mu \cos(h\xi))^2 + 4b^2\mu^2 \cos^2(h\xi) - 1}{1 + 2b\mu} = \frac{4b^2\mu^2 - 1}{(1 + 2b\mu)^2} = \frac{2b\mu - 1}{1 + 2b\mu} \leq 1.$

In addition, we do not want double roots on the unit circle. Double root occurs when $1 - 4b^2\mu^2 = 0$, but then $|g_{\pm}| \leq \frac{2b\mu|\cos(h\xi)|}{1 + 2b\mu} < 1$.

So we have stability for any value of μ . But how is that possible? The catch is in the consistency. In order for the scheme to be consistent we must have $k/h \rightarrow 0$, as we will now demonstrate.

Rewrite Du Fort–Frankel as

$$\frac{v_m^{n+1} - v_m^{n-1}}{2k} = b \frac{v_{m+1}^n - 2v_m^n + v_{m-1}^n}{h^2} - b \frac{v_m^{n+1} + v_m^{n-1} - 2v_m^n}{h^2}$$

then expand in Taylor to see that it approximates

$$u_t + \frac{k^2}{6}u_{ttt} = b\left(u_{xx} + \frac{h^2}{12}u_{xxxx}\right) - b\left(\frac{k^2}{h^2}u_{tt} + \frac{k^4}{12h^2}u_{tttt}\right).$$

Now think numerically. For hyperbolic systems we could (at best) hope for $\frac{k}{h} \approx 1$. However, if we use Du Fort–Frankel with such a timestep, $k = h$, the solution will not converge to the solution of $u_t = bu_{xx}$, but instead to the solution of $bu_{tt} + u_t = bu_{xx}$ (i.e., the solution to a wave equation). This was not the purpose of the exercise. So the scheme will only converge to the solution of $u_t = bu_{xx}$ if $\frac{k}{h} \rightarrow 0$. Even so the truncation error will be dominated by $b\frac{k^2}{h^2}u_{tt}$, which is not small unless $\frac{k}{h^2}$ is constant, but then we are back where we started—with the same restrictions as the ones for forward in time centered in space.

We, of course have two explicit schemes—backward in time, centered in space (which is $O(k + h^2)$ and dissipative) and Crank–Nicolson (which is $O(k^2 + h^2)$ and not dissipative if $\frac{k}{h}$ is constant.).