PRACTICE EXAM FOR EXAM 3

Instructions: Exam 3 will be in lecture on Monday, November 22. It will be closed book, closed notes, and calculators will not be allowed. You will have approximately 50 minutes for the exam. You should show all work, unless instructed otherwise; partial credit will be given only for work shown. When asked for justification, your argument need not be 100% rigorous, but it should be convincing.

Problem 1(15 points) For each angle $0 \leq \theta < 2\pi$, consider the matrix $A_\theta \in M_{2\times 2}(\mathbb{C})$.

$$A_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

(a)(10 points) For each θ other than $\theta = 0, \theta = \pi$, find the Jordan normal form of A_θ. You need not give a transition matrix which puts A_θ into the Jordan normal form, but do justify your answer.

Solution: We calculate the characteristic polynomial $c_{A_\theta}(X)$. It is

$$c_{A_\theta}(X) = (X - \cos\theta)^2 + \sin\theta^2 = \left(X^2 - 2\cos\theta + \cos\theta^2\right) + \sin\theta^2 = X^2 - 2\cos\theta X + 1.$$

Notice that if λ is a solution of $c_{A_\theta}(X)$, then we have $(\lambda - \cos\theta)^2 + \sin\theta^2 = 0$, i.e. $(\lambda - \cos\theta)^2 = (i \sin theta)^2$. So we must have $\lambda - \cos\theta = i\sin\theta$ or else $\lambda - \cos\theta = -i\sin\theta$. This gives two distinct eigenvalues, unless $\sin\theta = 0$. But, except for $\theta = 0, \theta = \pi$, we have that $\sin\theta \neq 0$ (in our range $0 \leq \theta < 2\pi$). Since we have two distinct eigenvalues, by our criterion for diagonalizability we conclude that A_θ is diagonaliable, and the Jordan normal form is

$$\begin{pmatrix} \cos\theta + i\sin\theta & 0 \\ 0 & \cos\theta - i\sin\theta \end{pmatrix}.$$

(b)(5 points) For $\theta = 0$ and for $\theta = \pi$, find the Jordan normal form of A_θ. You need not give a transition matrix which puts A_θ into the Jordan normal form, but do justify your answer.

Solution:
In the case that $\theta = 0$, we have

$$A_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

which is already diagonal. In the case that $\theta = \pi$, we have

$$A_\pi = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix},$$

which is also already diagonal.

Problem 2(20 points) Consider the matrix $B \in M_{3\times 3}(\mathbb{C})$:

$$B = \begin{pmatrix} -2 & 0 & 0 \\ 0 & -2 & -8 \\ 0 & 0 & 2 \end{pmatrix}.$$

Find the Jordan normal form of B. You need not give a transition matrix which puts B into the Jordan normal form, but do justify your answer.

Solution: Since B is triangular, its characteristic polynomial is simply $c_B(X) = (X + 2)(X + 2)(X - 2) = (X + 2)^2(X - 2)$. Since the algebraic multiplicity of 2 equals one, we know that in the Jordan normal form there will be one Jordan block associated to $\lambda = 2$ of size 1×1. Since the algebraic multiplicity of -2 is two, we know the generalized eigenspace of -2 has dimension two. In order to determine the Jordan block(s) associated to $\lambda = -2$, we compute the eigenspace of -2.

We form the matrix $B' = B + 2I_3$, i.e.

$$B' = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -8 \\ 0 & 0 & 4 \end{pmatrix}.$$

By Gauss-Jordan elimination, or by inspection, we see that the kernel has basis $(1, 0, 0)^\dagger, (0, 1, 0)^\dagger$. So the eigenspace of -2 equals the generalized eigenspace of -2. So B is diagonalizable and the Jordan normal form is simply:

$$\begin{pmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

Problem 3(15 points) Consider the matrix $C \in M_{8\times 8}(\mathbb{C})$ in Jordan normal form:

$$C = \begin{pmatrix} 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 \end{pmatrix}.$$

Answer each of the following questions. You need not justify your answer.

(a)(3 points) What is the characteristic polynomial of C?

Solution: Since C is triangular, its characteristic polynomial is easy to compute: $c_C(X) = (X - 3)^4(X - 2)^4$.

(b)(3 points) What is the characteristic polynomial of $C^\dagger$?

Solution: For all operators C, the characteristic polynomial of $C^\dagger$ equals the characteristic polynomial of C.

(c)(3 points) What is the dimension of the eigenspace E_3?

Solution: The eigenspace E_3 is the kernel of $C - 3I_8$. By inspection, this is spanned by the vectors corresponding to the columns of C where there is a single nonzero entry of 3, i.e. by $\mathbf{e}_3$ and $\mathbf{e}_4$. So E_3 has dimension two.

(d)(3 points) What is the dimension of the eigenspace E_2?

Solution: The eigenspace E_2 is the kernel of $C - 2I_8$. By inspection, this is spanned by the vectors corresponding to the columns of C where there is a single nonzero entry of 2, i.e. by $\mathbf{e}_6$ and $\mathbf{e}_8$. So E_2 has dimension two.

(e)(3 points) What is the dimension of $\ker((C - 3I_8)^2)$?

Solution: It isn't hard to show that

$$(C - 3I_8)^2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -2 & 1 \end{pmatrix}.$$

By inspection we conclude that the kernel is spanned by $\mathbf{e}_2, \mathbf{e}_3$ and $\mathbf{e}_4$. Thus $\ker((C - 3I_8)^2)$ has dimension three.

Problem 4(30 points) Let $n \geq 1$ be an integer. A matrix $E \in M_{n\times n}(\mathbb{C})$, $E = (a_{i,j})$, is a *stochastic matrix* if,

(i) for every $1 \leq i, j \leq n$, the entry $a_{i,j}$ is a nonnegative real number, and
(ii) for every $1 \leq i \leq n$, $\sum_{j=1}^{n} a_{j,i} = 1$, i.e., the sum of the entries in every column equals 1.

If E_1 and E_2 are stochastic $n \times n$ matrices, then also the product E_1E_2 is a stochastic $n \times n$ matrix. In particular if E is a stochastic matrix, then for every integer $n \geq 0$, E^n is also a stochastic matrix.

(a)(5 points) For every matrix $E \in M_{n\times n}(\mathbb{C})$, for every $\lambda \in \mathbb{C}$, for every λ-eigenvector for E, $v \in \mathbb{C}^n$, and for every integer $r \geq 0$, show that v is a λ^r-eigenvector for E^r. Your argument need not be 100% rigorous, but it should be convincing.

Solution: We prove this by induction on r. For $r = 1$ the statement is a tautology. Suppose, by way of induction, that the statement is known for $r - 1$. Then we have

$$A^r v = A(A^{r-1}v) = A(\lambda^{r-1}v) = \lambda^{r-1}Av = \lambda^{r-1}(\lambda v) = \lambda^r v.$$

So we conclude the result holds for r. Thus the result is proved by induction on r.

(b)(10 points) Let $E \in M_{n\times n}(\mathbb{C})$ be a stochastic matrix. Show that for every $\mathbf{v} = (c_1, \dots, c_n)^\dagger$ in $\mathbb{C}^n$, the sum,

$$|(E\mathbf{v})_1| + \cdots + |(E\mathbf{v})_n| = \sum_{i=1}^{n} |\sum_{j=1}^{n} a_{i,j}c_j|,$$

is at most $n(|c_1| + \cdots + |c_n|)$. In particular, if $\mathbf{v}$ is a nonzero λ-eigenvector, then $|\lambda| \leq n$.

Solution: As suggested, we consider the sum $\|Ev\|$. If v is an eigenvector of E with eigenvalue λ, then of course $\|Ev\| = \|\lambda v\| = |\lambda|\|v\|$. On the other hand, for ANY vector we have $\|Ev\| = \sum_{i=1}^{n} |\sum_{j=1}^{n} E_{i,j}v_j|)$. The absolute value of a sum of numbers is never greater than the sum of the absolute values of the numbers. Therefore $\|Ev\| \leq \sum_{i=1}^{n}\sum_{j=1}^{n} |E_{i,j}v_j|$. Of course we have $|E_{i,j}v_j| = |E_{i,j}||v_j|$. And since E is stochastic, each $|E_{i,j}|$ is at most 1. So we have $\|Ev\| \leq \sum_{i=1}^{n}\sum_{j=1}^{n} 1|v_j| = n\|v\|$. Therefore, if v is an eigenvector of E with eigenvalue λ, we conclude that

$$|\lambda|\|v\| = \|Ev\| \leq n\|v\|.$$

Since $\|v\|$ is positive, we can factor it out and we have $|\lambda| \leq n$ as required.

(c)(15 points) You may assume (a) and (b) above. For every stochastic matrix $E \in M_{n\times n}(\mathbb{C})$, for every eigenvalue $\lambda \in \mathbb{C}$ of E, show that $|\lambda| \leq 1$.

Solution: Suppose that v is an eigenvector of E with eigenvalue λ. Suppose, by way of contradiction, that $|\lambda| > 1$, say $|\lambda| = 1 + \epsilon$. Then for some large integer, say $M \geq \frac{n}{\epsilon}$, we have that $|\lambda|^M > n$. Now by part (a), we have that v is an eigenvector of E^M with eigenvalue λ^M. And we know that E^M is also a stochastic matrix. But then by part (b), we conclude that $|\lambda^M| \leq n$. Since $|\lambda^M| = |\lambda|^M$, we have a contradiction. This proves that $|\lambda| \leq 1$ as required.

EXTRA CREDIT:(5 points) Prove that for a stochastic matrix $E \in M_{n\times n}(\mathbb{C})$ has the eigenvalue $\lambda = 1$.

Solution:
We know that the eigenvalues of E are the same as the eigenvalues of $E^\dagger$. Consider the vector $v = (1, 1, \dots, 1)^\dagger$. We have $E^\dagger v$ is the vector whose ith coefficient is $\sum_{j=1}^{n} v_j E_{j,i} = \sum_{j=1}^{n} E_{j,i}$. Since E is stochastic, this sum is exactly 1. So we conclude that $E^\dagger v = v$, i.e. v is an eigenvector of $E^\dagger$ with eigenvalue 1. Therefore also E has an eigenvalue equal to one.

Problem 5(20 points) Let $n \geq 1$ be an integer. Let A be an $n \times n$ matrix with coefficients in a field $\mathbb{F}$ (which you may assume is $\mathbb{C}$ for simplicity). For every $P(X) = a_dX^d + \cdots + a_iX^i + \cdots + a_1X + a_0$, for every scalar $\mu \in \mathbb{F}$, the polynomial $P(\mu X)$ is defined by $a_d\mu^dX^d + \cdots + a_i\mu^iX^i + \cdots + a_1\mu X + a_0$ (this is the usual rule from high school algebra).

(a)(5 points) For every nonzero scalar $\mu \in \mathbb{F}$, show that

$$c_{\mu A}(X) = \mu^n c_A\left(\frac{1}{\mu}X\right).$$

Solution: By definition, $c_{\mu A}(X) = \det(XI_n - \mu A)$. Since the determinant is n-linear, we may factor out the μ and we get

$$c_{\mu A}(X) = \det\left(\mu\left(\frac{1}{\mu}XI_n - A\right)\right) = \mu^n \det\left(\frac{1}{\mu}XI_n - A\right) = \mu^n c_A\left(\frac{1}{\mu}X\right).$$

This is the equation we were to prove.

(b)(15 points) You may now assume (a). Show the equation,

$$(-1)^n c_A(-X)c_A(X) = c_{A^2}(X^2).$$

Hint: How can you factor $X^2 I_n - A^2$?

Solution: By part (a), we recognize $(-1)^n c_A(-X) = c_{-A}(X)$. Therefore we have

$$(-1)^n c_A(-X)c_A(X) = c_{-A}(X)c_A(X) = \det(XI_n + A)\det(XI_n - A).$$

Since the determinant is multiplicative, i.e. $\det(B)\det(C) = \det(BC)$ for any $B, C \in M_{n\times n}(\mathbb{F})$, we conclude that

$$\det(XI_n + A)\det(XI_n - A) = \det\left((XI_n + A)(XI_n - A)\right) = \\ \det\left(X^2 I_n + AXI_n + XI_n(-A) + A(-A)\right).$$

Gathering terms and simplifying, this is just

$$(-1)^n c_A(-X)c_A(X) = \det\left(X^2 I_n - A^2\right) = c_{A^2}(X^2).$$

This is the equation we were to prove.

EXTRA CREDIT(5 points) You may now assume (a) and (b) above. Write the characteristic polynomial of A as

$$c_A(X) = X^n - \text{trace}(A)X^{n-1} + a_{n-2}X^{n-2} + \dots$$

where $a_{n-2} \in \mathbb{F}$ is the coefficient of X^{n-2} in $c_A(X)$. Prove that

$$a_{n-2} = \frac{1}{2}\left(\text{trace}(A)^2 - \text{trace}(A^2)\right).$$

Solution: If we expand the equation $(-1)^n c_A(-X)c_A(X) = c_{A^2}(X^2)$ in highest powers of X, we have

$$(-1)^n\left((-X)^n - \text{trace}(A)(-X)^{n-1} + a_{n-2}(-X)^{n-2} + \dots\right) \times \\ \left(X^n - \text{trace}(A)X^{n-1} + a_{n-2}X^{n-2} + \dots\right) = X^{2n} - \text{trace}A^2 X^{2n-2} + \dots.$$

Gathering terms and simplifying, the left-hand side becomes

$$(-1)^n\left((-1)^n X^{2n} + ((-1)^n\text{trace}(A) - (-1)^n\text{trace}(A))X^{2n-1} + \\ ((-1)^n a_{n-2} - (-1)^n\text{trace}(A)^2 + (-1)^n a_{n-2})X^{2n-2} + \dots\right).$$

Cancelling powers of $(-1)^n$, this becomes

$$X^{2n} + 0X^{2n-1} - (\text{trace}(A)^2 - 2a_{n-2})X^{2n-2} + \cdots = X^{2n} - \text{trace}(A^2)X^{2n-2} + \ldots.$$

Equating coefficients of X^{2n-2} on the left-hand side and the right-hand side, we conclude that $\text{trace}(A^2) = \text{trace}(A)^2 - 2a_{n-2}$. Therefore we have

$$a_{n-2} = \frac{1}{2}\left(\text{trace}(A)^2 - \text{trace}(A^2)\right).$$